'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2021

CC6-MATHEMATICS

GROUP THEORY-I

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) Describe all the permutations on the set $\{x, y, z\}$ and find their respective orders. 3

(b) Find all homomorphisms from the group $(\mathbb{Z}_6, +)$ to $(\mathbb{Z}_4, +)$. 3

(c) Find the number of elements of order 5 in the group $(\mathbb{Z}_{30}, +)$ 3

(d) Let H be a subgroup of the group G. Show that $\{aH : a \in G\}$ forms a partition of G. 3

(e) "Commutativity of a factor group of the group G does not imply commutativity of G". Justify the statement. 3

(f) Prove that Cosets of a subgroups of the group are mutually exclusive and exhaustive. 3

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) If H be a subgroup of a cyclic group G, then the quotient group G/H is cyclic. 6

Is converse of this result true? Justify your answer.

(b) Show that the multiplicative group $\mathbb{R}^*$ of all non-zero real numbers is the internal direct product of the set of all positive real numbers $\mathbb{R}^+$ and the set $T = \{1, -1\}$. 4+2

Also find the number of elements of order 5 in $\mathbb{Z}_{15} \times \mathbb{Z}_5$.

(c) Let $G = S_3$ and G' be the multiplicative group $\{1, -1\}$. Let $\phi : G \to G'$ is defined by 3+2+1

$$\phi(a) = 1 \text{ if } a \text{ is an even permutation}$$
$$= -1 \text{ if } a \text{ is an odd permutation.}$$

Show that ϕ is an epimorphism. Also find $\ker\phi$ and hence determine a normal subgroup of S_3.

(d) Using group theory prove that $(1320)^6 \equiv 1(\text{mod } 7)$. 6

(e) Find all cosets of the subgroup $\langle 4\rangle$ of the group $\mathbb{Z}_{12}$. 6

(f) (i) Let $\varphi:(G,\circ)\to(G',*)$ be a group homomorphism. Prove that for $a\in G$, $\varphi(a^n)=\{\varphi(a)\}^n$ where $n\in\mathbb{Z}$. 3

(ii) Let G and G' be two groups with $o(G)=10$ and $o(G')=6$. Does there exists a homomorphism of G onto G'? Justify your answer. 3

GROUP-C

Answer any *two* questions 12×2 = 24

3. (a) Let G be a group of all non-zero complex numbers under multiplication and $F\left\{\begin{pmatrix} a & b \\ -b & a\end{pmatrix}: a,b\in\mathbb{R},\, a^2+b^2\neq 0\right\}$ be a group under matrix multiplication. Show that $G\cong F$. 6

(b) Let $a=\begin{pmatrix}1 & 2 & 3\\ 3 & 2 & 1\end{pmatrix}$, $b=\begin{pmatrix}1 & 2 & 3\\ 1 & 3 & 2\end{pmatrix}$. Find the solution of $ax=b$ in S_3. 6

4. (a) Let $(G,*)$ be a group and $a*b*a^{-1}=b^2$ for $a,b\in G$. If $o(a)=3$ and $b\neq e_G$, then find $o(b)$. 2

(b) Prove that a cyclic group of finite order n has one and only one subgroup of order d for every positive divisor d of n. 5

(c) Show that any two left cosets of a subgroup H of the group G are either identical or they have no common element. 5

5. (a) Let H and K be two finite cyclic groups of order m and n respectively. Prove that the direct product $H\times K$ is cyclic group iff $\gcd(m,n)=1$. 6

(b) Let H be a subgroup of the group G and $[G:H]=2$. Show that for every $x\in G$, $x^2\in H$. 3

(c) Let G be a non-abelian group of order p^3, where p is a prime number. Find the order of centre of the group G. 3

6. (a) Let H and K be two normal subgroups of the group G such that $H\cap K=\{e\}$. Prove that $hk=kh$ for all $h\in H$ and $k\in K$. 6

(b) Show that $\mathbb{Z}_9$ is not a homomorphic image of $\mathbb{Z}_{16}$. 6

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