'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2022

GE2-P2-MATHEMATICS

Time Allotted: 2 Hours   Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE4-I, MATHGE4-II, MATHGE4-III, MATHGE4-IV & MATHGE4-V. The candidates are required to answer any *one* from the *five* courses. Candidates should mention it clearly on the Answer Book.

MATHGE4-I

CAL. GEO. AND DE.

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) If $y = \tan^{-1}\frac{x}{a}$, then find y_n. 3

(b) Evaluate: $\lim\limits_{x\to 0}\dfrac{\sin x - x + \frac{x^3}{6}}{x^5}$ 3

(c) Find the equation of the circle which contains the point of intersection of the lines $x+3y-6=0$ and $x-2y-1=0$, and centre at origin. 3

(d) Identify the locus of the equation $x^2+y^2+6x-4y+9=0$. 3

(e) Obtain a reduction formula for $\int x^n e^{ax}dx$ and hence evaluate $\int x^3 e^{ax}dx$. 2+1

(f) Solve: $\frac{dy}{dx}+2xy = x^2+y^2$ 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) If $y = a\cos(\log x)+b\sin(\log x)$, show that $x^2y_{n+2}+(2n+1)xy_{n+1}+(n^2+1)y_n=0$. 6

(b) Show that the points of inflexion of the curve $y^2=(x-a)^2(x-b)$ lie on the line $3x+a=4b$. 6

(c) Find the volume of the solid generated by the revolution of the cardioide $r=a(1+\cos\theta)$ about the polar axis. 6

(d) A plane passes through a fixed point $(p,\, q,\, r)$ and cuts the axes A, B, C. Find the locus of the centre of the sphere $OABC$. 6

(e) Solve: $(2xy^4e^y+2xy^3+y)\,dx+(x^2y^4e^y-x^2y^2-3x)\,dy=0$ 6

(f) Find the equation of the parabola whose focus is at $(3,-4)$ and directrix as the line $x+2y-2=0$. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Find an integrating factor of the form $x^p y^q$ and solve the equation 6

$$(4xy^2 + 6y)\,dx + (5x^2y + 8x)\,dy = 0$$

(b) Solve, by reducing to Clairaut's form by putting $x^2 = u$ and $y^2 = v$, the differential equation $(px - y)(x - py) = 2p$. 6

4. (a) Find the asymptotes of the curve $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 = 2$. 6

(b) Find the envelope of circles whose centre lie on the rectangular hyperbola $xy = c^2$ and pass through its centre. 6

5. (a) Reduce the equation $\dfrac{dy}{dx} = 1 - x(y - x) - x^3(y - x)^3$ to a linear form and hence solve it. 6

(b) Find the singular solution of the differential equation satisfied by the family of curves $c^2 + 2cy - x^2 + 1 = 0$, where c is a parameter. 6

6. (a) Find the length of the arc of the curve 6

$$x = c\sin 2\theta(1 + \cos 2\theta) \quad , \quad y = c\cos 2\theta(1 - \cos 2\theta)$$

from the origin to any point.

(b) Find the area of the curve $a^2y^2 = a^2x^2 - x^4$. 6

MATHGE4-II

ALGEBRA

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Find two integers u and v satisfying $54u + 24v = 30$. 3

(b) Find the value of $\sqrt[3]{i} + \sqrt[3]{-i}$, where $\sqrt[3]{z}$ is the principal cube root of z. 3

(c) Find the rank of the matrix, 3

$$\begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}$$

(d) Find the remainder when 3^{36} is divided by 77. 3

(e) Show that the mapping $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(n) = \begin{cases} \dfrac{n}{2} & , \text{ if } n \text{ is even} \\ -\dfrac{n-1}{2} & , \text{ if } n \text{ is odd} \end{cases}$ is invertible. 3

(f) Without solving, state the nature of roots of the equation $x^7 - 3x^3 - x + 1 = 0$. 3

GROUP-B

2. Answer any ***four*** questions from the following: $6\times4=24$

(a) For $a, b, c > 0$, show that $(ab+bc+ca)(ab^{-1}+bc^{-1}+ca^{-1}) \geq (a+b+c)^2$. 6

(b) Find all eigen values and corresponding eigen vectors of the matrix. 6

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

(c) A relation ρ on the set $\mathbb{N}$ is given by "$\rho = \{(a, b) \in \mathbb{N}\times\mathbb{N}:\ a \mid b\}$". Examine if ρ is (i) reflexive, (ii) symmetric, (iii) transitive. 6

(d) Solve the equation $x^4-4x^3-4x^2-4x-5=0$, given that two roots α, β are connected by the relation $2\alpha+\beta=3$. 6

(e) (i) Prove that if $a \equiv b \pmod{m}$, then $a^n \equiv b^n \pmod{m}$ for all positive integer n. 2

(ii) Prove that $3^{2n}-8n-1$ is divisible by 64. 4

(f) If $\tan^{-1}(x+iy) = \alpha+i\beta$, where x, y, α, β are real and $(x, y) \neq (0, \pm1)$, then prove that 6

(i) $x^2+y^2+2x\cot 2\alpha = 1$

(ii) $x^2+y^2+1-2y\coth 2\beta = 0$.

GROUP-C

Answer any *two* questions from the following $12\times2=24$

3. (a) Verify Cayley-Hamilton theorem for the matrix 7

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Hence find A^{-1} and A^9.

(b) If $x+\frac{1}{x} = 2\cos\theta$, then show that for any positive integer n, $x^n+\frac{1}{x^n} = 2\cos n\theta$, $x^n-\frac{1}{x^n} = \pm\, 2i\sin n\theta$ and $\frac{x^{2n}-1}{x^{2n}+1} = \pm\, i\tan n\theta$. 5

4. (a) Show that the relation $a \equiv b \pmod{5}$ is an equivalence relation. 6

(b) Show that $1!\,3!\,5!\cdots(2n-1)! > (n!)^n$ for all $n \in \mathbb{N}$. 6

5. (a) Solve the equation $4x^4+20x^3+35x^2+24x+6=0$, whose roots are in A.P. 6

(b) Find the product of all values of $(1+i)^{4/5}$. 3

(c) Show that the eigen values of a real symmetric matrix are all real. 3

6. (a) Solve the system of linear equations given by: 6

$$2x+4y+6z+4w=4$$
$$2x+5y+7z+6w=3$$
$$2x+3y+5z+2w=5$$

(b) Prove by induction, $1^2+2^2+3^2+\cdots\cdots+n^2=\frac{1}{6}n(n+1)(2n+1)$. 6

MATHGE4-III

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Solve the equation $(1-x^2)\,dy=2y\,dx$, when $x=2$, $y=1$.

(b) Find $\frac{1}{D^2-4}(\cos^2 x)$.

(c) Show that $\sin x$, $\cos x$, $\sin 2x$ are linearly dependent.

(d) Construct the differential equation from the relation $V=\frac{A}{r}+B$ by eliminating the arbitrary constant A and B.

(e) Show that the vectors $(i-2j+3k)$, $(-2i+3j-4k)$, $(-j+2k)$ are co-planar.

(f) If $\vec{a}=2t^2\hat{i}+3(t-1)\hat{j}+4t^2\hat{k}$ and $\vec{b}=(t-1)\hat{i}+t^2\hat{j}+(t-2)\hat{k}$, find $\int_0^2(\vec{a}\cdot\vec{b})\,dt$.

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) If $y_1,\ y_2,\ \cdots\cdots,\ y_n$ be n solutions of the differential equation 6

$$a_n\frac{d^n y}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots\cdots+a_1\frac{dy}{dx}+a_0 y=0$$

(where $a_0, a_1, \ldots, a_n$ are all constants) then show that $y=\lambda_1 y_1+\lambda_2 y_2+\ldots+\lambda_n y_n$ will be another solution of the equation for any scalars $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$.

(b) Solve: $y''-4y'-5y=xe^{-x}$, $y(0)=0$, $y'(0)=0$ 6

(c) (i) Solve: 3+3

$$\frac{dx}{dt}+2x-3y=t$$
$$\frac{dy}{dt}-3x+2y=e^{2t}$$

(ii) Apply method of undetermined coefficient to solve $\frac{d^2y}{dx^2}-2\left(\frac{dy}{dx}\right)-3y=2e^x$.

(d) Solve: $(D^4-8D)y=x^2+e^{2x}$, $D\equiv\frac{d}{dx}$ 6

(e) (i) If $\vec{r}=(2x^2y-x^4)\hat{i}+(e^{xy}-y\sin x)\hat{j}+(x^2\cos y)\hat{k}$, show that $\frac{\partial^2\vec{r}}{\partial x\partial y}=\frac{\partial^2\vec{r}}{\partial y\partial x}$. 3+3

(ii) Calculate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}=y\hat{i}+z\hat{j}+x\hat{k}$ and C is the circle $x^2+y^2=1$, $z=0$.

(f) Prove that the necessary and sufficient condition for a vector valued function $\vec{a}(t)$ to be of constant magnitude is $\vec{a}\cdot\frac{d\vec{a}}{dt}=0$. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Solve: $(D^2-3D+2)y=\sin 3x$ 4

(b) Solve: $x^2y_2+xy_1-4y=0$ 4

(c) Find $\lim_{t\to 0} f(t)$ given that $f(t)=\dfrac{\sin t+t}{3t}\,\hat{i}+e^{2t}\,\hat{j}+\sin(t-n)\hat{k}$. 4

4. (a) Solve the given differential equation $(x^3D^3+2x^2D^2+2)y=10\left(x+\dfrac{1}{x}\right)$. 6

(b) Solve: $(D^2+6D+8)y=(e^{2x}+1)^2$ 6

5. (a) If $\vec{x}=(a\cos t)\hat{i}+(a\sin t)\hat{j}+(at\tan\alpha)\hat{k}$, then show that 6

$$\left[\frac{d\vec{x}}{dt}\;\frac{d^2\vec{x}}{dt^2}\;\frac{d^3\vec{x}}{dt^3}\right]=a^3\tan\alpha$$

(b) Evaluate $\oint_T \vec{F}\cdot d\vec{r}$, where $\vec{F}=(2x+y^2)\hat{i}+(3y-4x)\hat{j}$ and T is the triangle where vertices are (0, 0), (2, 0), (2, 1) taking this order. 6

6. (a) If $\vec{V}=xy\hat{i}-z^2\hat{j}+xyz\hat{k}$ and C be a curve given by $\vec{r}=t\hat{i}+t^2\hat{j}+t^3\hat{k}$, from (0, 0, 0) to (1, 1, 1), then calculate 6

$$\int_C \vec{V}\cdot d\vec{r}$$

(b) If the position vector of a moving particle at any time t is given by $\vec{r}=\sin t\,\hat{i}+\cos t\,\hat{j}+2t\,\hat{k}$, then show that the velocity of the particle has a constant magnitude. 3

(c) If $\vec{a}=2t^2\hat{i}+3(t-1)\hat{j}+4t^2\hat{k}$ and $\vec{b}=(t-1)\hat{i}+t^2\hat{j}+(t-2)\hat{k}$, find $\int_0^2(\vec{a}\cdot\vec{b})\,dt$. 3

MATHGE4-IV

GROUP THEORY

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Let $(G,\ \circ)$ be a group and $a,b\in G$. If $a^2=e$ and $a\circ b^2\circ a=b^3$, then prove that $b^5=e$. 3

(b) Is the union of two subgroups of a group G also a subgroup of G? Explain it. 3

(c) If b be an element of a group and $O(b)=20$, find the order of the element b^{15}. 3

(d) Show that any group of order three is cyclic. 3

(e) Prove that the group $SL(2,\mathbb{R})$ is a normal subgroup of the group $GL(2,\mathbb{R})$. 3

(f) Write down the elements in the group S_3. 3

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) In a group *G*, for all $a, b \in G$, $(ab)^n = a^n b^n$ holds for three consecutive integers *n*. Prove that the group is abelian. 6

(b) Prove that a non-abelian group of order 8 must have an element of order 4. 6

(c) (i) Show that intersection of two subgroups of a group is also a subgroup of the group. 3

(ii) If *G* is a commutative group then prove that $H = \{a^2 : a \in G\}$ is a subgroup of *G*. 3

(d) Show that $(\mathbb{Z}, *)$ is a group where $*$ is defined by $a * b = a + b - 1 \ \ \forall\, a, b \in \mathbb{Z}$. Is it a commutative group? 4+2

(e) Prove that every subgroup of a cyclic group is cyclic. 6

(f) Prove that a group $(G, \cdot)$ is abelain if and only if $(a \cdot b)^{-1} = a^{-1} \cdot b^{-1} \quad \forall a, b \in G$. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Let *H* be a subgroup of a group *G*. The relation ρ defined on *G* by "$a\rho b$ iff $a^{-1}b \in H$" for $a, b \in G$ is an equivalence relation on *G*. 6

(b) Prove that every cyclic group is abelian. Is the converse true? — Justify. 4+2

4. (a) If *H* be a subgroup of a commutative group *G*, then prove that the quotient group G/H is commutative. Is the converse true? — Justify. 3+3

(b) If *H* and *K* are subgroups of a group $(G, \cdot)$, then show that *HK* is a subgroup of $(G, \cdot)$ if and only if $HK = KH$. 6

5. (a) Show that in a group $(G, *)$ 3+3

(i) the inverse of each element is unique.

(ii) the equation $a * x = b$ has a unique solution $\forall\, a, b \in G$.

(b) Let $M = \left\{ \begin{pmatrix} x & y \\ x & y \end{pmatrix} : x, y \in \mathbb{R} \text{ and } x + y \neq 0 \right\}$. Check whether *M* forms a group with respect to multiplication. 6

6. (a) Find all homomorphism from $(\mathbb{Z}_8, +_8)$ to $(\mathbb{Z}_6, +_6)$. 6

(b) Let $G = S_3$, $G' = (\{1, -1\}, \cdot)$ and $\phi : G \to G'$ is defined by 2+4

$$\phi(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ be an even permutation in } S_3 \\ -1 & \text{if } \alpha \text{ be an odd permutation in } S_3 \end{cases}$$

Determine $\ker\phi$. Deduce that A_3 is a normal subgroup of S_3.

MATHGE4-V

NUMERICAL METHODS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) Find the relative error in the computation of $x-y$ for $x=12.05$ and $y=8.02$ having absolute error $\Delta x = 0.005$ and $\Delta y = 0.001$. 3

(b) Write down the order of convergence of the following methods: 1+2 = 3

(i) Newton-Raphson method

(ii) Gauss-Jacobi iteration method.

(c) State three differences between direct and iterative method. 3

(d) Explain the Geometrical interpretation of trapezoidal rule. 3

(e) Define 'degree of precession' of a quadrature formula and find the degree of precession of Trapezoidal rule. 2+1 = 3

(f) Write down the number of significant figures in the following: 1+1+1 = 3

5.398, 0.000538, 9.123

GROUP-B

2. Answer any ***four*** questions from the following: 6×4 = 24

(a) Use the method of bisection to compute a real root of $x^3 - 4x - 9 = 0$ between 2 and 3 and correct upto four significant figures. 6

(b) Use Gauss-Jacobi method to solve: 6

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1$$

(c) Explain the principle of propagation of errors and explain how it affects numerical computation. 6

(d) Find $y(4.4)$ by Euler's modified method, taking $h = 0.2$ from the differential equation. 6

$$\frac{dy}{dx} = \frac{2-y^2}{5x} \quad , \quad y = 1 \text{ when } x = 4$$

(e) Find the value of $\int_0^1 \frac{dx}{1+x^2}$, taking 5-sub-intervals, by Trapezoidal rule, correct to 5 significant figures. 6

(f) Evaluate the missing terms in the following table: 6

x	0	1	2	3	4	5
$f(x)$	0	–	8	15	–	35

GROUP-C

Answer any *two* questions from the following 12×2 = 24

3. (a) Use Runge-Kutta method of order two to find $y(0\cdot 1)$ and $y(0\cdot 2)$ correct upto four decimal places, when given $\frac{dy}{dx} = y - x$, $y(0) = 2$. 6

   (b) What is interpolation? Establish the Lagrange's interpolation polynomial formula. 6

4. (a) Use Picard's method to compute $y(0\cdot 1)$ from the differential equation: 6

$$\frac{dy}{dx} = x + y \quad ; \quad y = 1 \text{ when } x = 0$$

   (b) Explain Euler's method for solving first order differential equation of the form: 6

$$\frac{dy}{dx} = f(x,\ y)$$

5. (a) Use Gauss-Seidal method to solve the following system of equations: 6

$$8x_1 + 2x_2 - 2x_3 = 8$$
$$x_1 - 8x_2 + 3x_3 = 4$$
$$2x_1 + x_2 + 9x_3 = 12$$

   (b) Find the condition of convergence of fixed point iteration method. 6

6. (a) Find a positive root of $x^2 + 2x - 2 = 0$ by Newton-Raphson method, correct upto two significant figures. 6

   (b) A function $f(x)$ defined on [0, 1] is such that $f(0) = 0$, $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 2$. Find the interpolating polynomial which approximate equal to $f(x)$. 6

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