'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2022

CC9-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-I

Time Allotted: 2 Hours Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. Find a basis and dimension of the subspace W of $\mathbb{R}^3$, where $W=\{(x,\,y,\,z)\in\mathbb{R}^3:\ x+y+z=0\}$. 3

2. Let R be a finite ring with n elements and S be a subring of R containing m elements. Prove that m is a divisor of n. 3

3. Consider the ring $\mathbb{Z}\times\mathbb{Z}$ under component-wise addition and multiplication. Show that the set $S=\{(a,\ 0):\ a\in\mathbb{Z}\}$ is a subring of $\mathbb{Z}\times\mathbb{Z}$ having unity different from that of $\mathbb{Z}\times\mathbb{Z}$. 3

4. Prove that the set $S=\{(1,0,1),(0,1,1),(1,1,0)\}$ is a basis of $\mathbb{R}^3$. 3

5. Let D be an integral domain and $a,b\in D$. If $a^5=b^5$ and $a^8=b^8$, then prove that $a=b$. 3

6. Is the ring $2\mathbb{Z}$ isomorphic to the ring $5\mathbb{Z}$? — Justify. 3

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. (a) Determine k, so that the set $S=\{(k,\,1,\,1),\,(1,\,k,\,1),\,(1,\,1,\,k)\}$ is linearly independent in $\mathbb{R}^3$. 3

   (b) If T is linear, then $\ker T=\{\theta\}$ iff T is injective. 3

8. (a) Let T be a linear mapping on the real vector space P_4 defined by, $T(p(x)) = x\frac{d}{dx}(p(x))$, $p(x) \in P_4$. Determine the matrix of T relative to the standard basis of P_4. 3

(b) Find the dimension of the subspace S of $\mathbb{R}^3$ defined by, 3

$$S = \{(x, y, z) \in \mathbb{R}^3 : \quad x + 2y = z, \;\; 2x + 3z = y\}$$

9. Let R and R' be two rings and $\phi : R \to R'$ be an onto homomorphism. If I is an ideal of R, show that $\phi(I)$ is also an ideal of R'. Will this statement still be true if ϕ is any arbitrary homomorphism from R to R'? 4+2

10. Let U and W be two subspaces of a finite dimensional vector space V. Show that $\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W)$. 6

11. Prove that a finite integral domain is a field. 6

12.(a) Let 3

$$R = \{a + b\sqrt{3} : \; a, b \in \mathbb{Z}\},$$
$$S = \{2a + 2b\sqrt{3} : \; a, b \in \mathbb{Z}\} \text{ and}$$
$$T = \{4a + 2b\sqrt{3} : \; a, b \in \mathbb{Z}\}$$

Show that T is an ideal of S, but not an ideal of R.

(b) Find the units in the integral domain $\mathbb{Z}$ [i]. 3

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Prove that the ring $(\mathbb{Z}_n, +, \cdot)$ is an integral domain iff n is a prime. 4

(b) Find $\dim(U \cap V)$, where U and V are subspaces of $\mathbb{R}^4$ given by 4

$$U = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; x_1 + x_2 + x_3 + x_4 = 0\},$$
$$V = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \; 2x_1 + x_2 - x_3 + x_4 = 0\}$$

(c) Let R be a ring with unity and the left ideals of R are only the null ideal and R itself. Show that R is a skew field. 4

14.(a) Extend the set $\{(1, 1, 1, 1), (1, -1, 1, -1)\}$ to a basis of $\mathbb{R}^4$. 4

(b) Give an example of a subring which is not an ideal. 2

(c) The matrix of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ relative to the ordered bases $\{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}$ of $\mathbb{R}^3$ and $\{(1, 0), (1, 1)\}$ of $\mathbb{R}^2$ is $\begin{pmatrix} 1 & 2 & 4 \\ 2 & 1 & 0 \end{pmatrix}$. 6

Find the matrix of T relative to the ordered bases $\{(1, 1, 0), (1, 0, 1), (0, 1, 1)\}$ of $\mathbb{R}^3$ and $\{(1, 1), (0, 1)\}$ of $\mathbb{R}^2$.

15.(a) Does there exist an epimorphism from the ring $\mathbb{Z}_{24}$ onto the ring $\mathbb{Z}_7$? 3

(b) Let I be an ideal of a ring R. Prove that if $\mathbb{R}$ is a commutative ring with unity, then so is R/I. If R has no divisor of zero, is the same necessarily true for R/I. 6

(c) Let α, β, γ be three vectors in a vector space V, so that $\alpha+\beta+\gamma=\theta$. Show that $L(\{\alpha, \beta\}) = L(\{\beta, \gamma\}) = L(\{\gamma, \alpha\})$ 3

16.(a) Find a basis and determine the dimension of the set of all 2×2 real skew symmetric matrices. 4

(b) Show that the rings $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic. 2

(c) Let R be the ring of all real valued continuous functions on $[0, 1]$. A mapping $\phi: R \to \mathbb{R}$ is defined by $\phi(f) = f\left(\frac{1}{2}\right)$ $\forall f \in R$. Show that ϕ is an onto homomorphism. Determine $\ker\phi$. Prove that $R/\ker\phi \simeq \mathbb{R}$. 6

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