'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours Part-III Examination, 2022

MATHEMATICS

PAPER-X

REAL ANALYSIS, INTEGRAL CALCULUS

NEW SYLLABUS

Time Allotted: 2 Hours | Full Marks: 50

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer Question No. 1 and any *two* from the rest

1. (a) Find the radius of convergence of the power series 2

$$x+\frac{1!}{2^2}x^2+\frac{2!}{3^3}x^3+\frac{3!}{4^4}x^4+\cdots\cdots$$

(b) State Cauchy-Hadamard theorem. 1

(c) Prove that the series $\sum\frac{1}{n^3+n^4x^2}$ is uniformly convergent for all real x. 2

2. (a) State and prove Heine-Borel theorem. 6

(b) If a series of uniformly continuous functions is uniformly convergent, show that the limit function is also uniformly continuous. 4

3. (a) A function f is defined on $\mathbb{R}$ by 6

$$f(x)=\begin{cases} -x^2 & , \quad x\le 0 \\ 5x-4 & , \quad 0<x\le 1 \\ 4x^2-3x & , \quad 1<x\le 2 \\ 3x+4 & , \quad x\ge 2 \end{cases}$$

Examine f for continuity at $x=0,1,2$. Also discuss the kind of discontinuity, if any.

(b) The space ($\mathbb{R}$, d) is not compact, where $\mathbb{R}$ is the set of real numbers and d is the usual metric. 2

(c) State Cantor's intersection theorem. 2

4. (a) Use Lagrange's method of undetermined multipliers to find the length of the greatest chord of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ passing through the origin. 6

(b) Show that the sequence $\{f_n\}$, converges pointwise to zero on $[0,1]$, where $f_n(x)=nxe^{-nx^2}$, $n=1,\ 2,\ 3,\ \cdots\cdots$ 4

5. (a) Show that closed subset of compact metric space is compact. 5

(b) Show that f is continuous for $x>0$, where $f(x)=e^{-x}+2e^{-2x}+3e^{-3x}+\cdots\cdots$ 5

Also evaluate $\int\limits_{\log 2}^{\log 3} f(x)\,dx$.

GROUP-B

Answer Question No. 6 and any *two* from the rest

6. (a) Change the order of integration of the integral $\int\limits_0^2 \int\limits_{-y}^{\sqrt{y}} (1+x+y)\ dx\,dy$. 2

(b) Test the convergence of the integral $\int\limits_0^\infty \sin x^2\, dx$. 2

(c) Prove that $\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\pi\sec\frac{n\pi}{2}$, $-1<n<1$. 1

7. (a) Show that the integral $\int\limits_0^{\pi/2} \log\sin x\ dx$ is convergent and hence evaluate it. 6

(b) Examine the convergence of 2+2

(i) $\int\limits_0^1 \frac{dx}{x^2}$, (ii) $\int\limits_0^1 \frac{dx}{\sqrt{1-x}}$

8. (a) If f is bounded and integrable on $[-\pi,\pi]$ and a_n, b_n are its Fourier coefficients then prove that $\sum\limits_{n=1}^{\infty}(a_n^2+b_n^2)$ converges. 6

(b) Compute the surface area of the sphere $x^2+y^2+z^2=a^2$. 4

9. (a) Expand $(x+x^2)$ in Fourier series in $-\pi<x<\pi$ and deduce that 6

$$\frac{\pi^2}{4}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots$$

(b) Discuss the convergence of $\int\limits_0^\infty \frac{x^\alpha}{1+x^\beta\sin^2 x}\,dx$. 4

10.(a) Using the method of differentiation under the integral sign (arbitrary parameter), show that $\int\limits_0^{\pi/2} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\frac{d\theta}{\sin\theta}=\pi\sin^{-1}\left(\frac{b}{a}\right)$, $a>b\geq 0$. 6

(b) If R is the region in the xy plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$, prove that $\iint\sqrt{x^2+y^2}\,dx\,dy=\frac{14}{3}\pi$. 4

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