UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 3rd Semester Examination, 2020

GE-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE1, MATHGE2, MATHGE3, MATHGE4 and MATHGE5. Candidates are required to answer any *one* from the *five* MATHGE courses and they should mention it clearly on the Answer Book.

MATHGE1

Calculus, Geometry and DE

GROUP-A

Answer *all* questions $2\times 6 = 12$

1. (a) Find the n-th derivative of $\dfrac{x^2}{x-1}$. 2

(b) Evaluate $\lim\limits_{x\to 0}\left(\dfrac{\tan x}{x}\right)^{1/x}$. 2

(c) Find $\displaystyle\int_0^{\pi/2}\sin^7 x\,dx$. 2

(d) Find the asymptotes of the curve $y = 2\sqrt{x^2+4}$. 2

(e) Prove that $y = x^4$ is concave upwards at the origin. 2

(f) Find the Envelope of the lines $\dfrac{x}{a}+\dfrac{y}{b}=1$, where the parameters a, b are connected by the equation $a^n + b^n = c^n$, c, n are the given constants. 2

GROUP-B

Answer *all* questions $5\times 4 = 20$

2. (a) If $x = \dfrac{\sin^{-1}t}{\sqrt{1-t^2}}$, $|t|<1$, then show that 5

$$(1-t^2)x_{n+2} - (2n+3)x_{n+1} - (n+1)^2 x_n = 0$$

(b) Find the surface area of the region bounded by the plane $z = x+y$ and the paraboloid $cz = x^2+y^2$. 5

(c) If $I_n = \int_0^{\pi/4} \tan^n x\, dx$, show that $I_{n+1} + I_{n-1} = \frac{1}{n}$. 5

(d) Show that the volume of the solid formed by revolving one loop of the curve $r^2 = a^2 \cos 2\theta$ about the $\theta = \frac{\pi}{2}$ is $\frac{\pi^2 a^3}{4\sqrt{2}}$. 5

GROUP-C

Answer *all* questions 7×4 = 28

3. (a) (i) If $\lim\limits_{x \to 0} \frac{\sin 2x + a \sin x}{x^3}$ is finite, find a and the value of the limit. 3

(ii) Find the area of a loop of the curve 4

$$a^4 y^2 = x^4 (a^2 - x^2)$$

(b) (i) Show that the four asymptotes of the curve 4

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2 y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve in eight points which lie on the circle $x^2 + y^2 = 1$.

(ii) Find the point of inflexion of the curve $y^2 = x(x+1)^2$. 3

(c) (i) If $\sinh x = \tan\theta$, express $\cosh x$ and $\tanh x$ in terms of θ and show that 3

$$x = \log_e \left\{ \tan\left(\frac{\pi}{4} + \frac{\theta}{2} \right) \right\}$$

(ii) If $(x+y)^{m+n} = x^m y^n$, prove that $\frac{dy}{dx}$ is independent of m and n. 4

(d) (i) Find the length of the arc of the parabola $y^2 = 16x$ measured from the vertex to an extremity of its latus rectum. 4

(ii) If $y = \sin(m \cos^{-1} \sqrt{x})$, then prove that 3

$$\lim_{x \to 0} \frac{y_{n+1}}{y_n} = \frac{4n^2 - m^2}{4n + 2}$$

MATHGE2

Algebra

GROUP-A

1. Answer ***all*** questions from the following: 2×6 = 12

(a) If the sum of the two roots of the equation 2

$$x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

is zero, prove that $a_1 a_2 = a_3$.

(b) Apply Descarte's rule of signs to find the nature of roots of $x^4 + 18x^2 + x - 12 = 0$. 2

(c) Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$ when the roots are in H. P. 2

(d) If $A=\begin{pmatrix} 1 & 0 & -1 \\ -1 & 2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$, then find the rank of A. 2

(e) Check whether $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x,\ y,\ z)=(x+2,\ y+2,\ z+2)$ is a linear map. 2

(f) If A and B be invertible matrices of same order then prove that $(AB)^{-1}=B^{-1}A^{-1}$. 2

GROUP-B

2. Answer ***all*** the questions from the following: $5\times 4=20$

(a) Solve by matrix method the following system of equations: 5

$$x+y+z=3$$
$$2x-y+z=1$$
$$x+3y-z=1$$

(b) Solve the equation by Cardan's Method: 5

$$x^3-12x+8=0$$

(c) Find the matrix A if $A^{-1}=\begin{pmatrix} 3 & -1 & 1 \\ 1 & -2 & 3 \\ 3 & -3 & 4 \end{pmatrix}$. 5

(d) If $\alpha,\ \beta,\ \gamma$ and δ are the roots of the equation $x^4+px^3+qx^2+rx+s=0$ such that $\alpha\beta+\gamma\delta=0$, then prove that $p^2s+r^2-4qs=0$. 5

GROUP-C

3. Answer ***all*** the questions from the following: $7\times 4=28$

(a) (i) Show that the vectors $(0,2,-4),(1,-4,3)$ and $(1,-2,-1)$ of $\mathbb{R}^3$ are linearly dependent. 3+4

(ii) Find the matrix of the linear transformation $T:\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by

$$T(x,\ y,\ z)=(2x+y+3z,\ 3x-y+z,\ -4x+3y+z)$$

(b) (i) The roots of the equation $x^3+px^2+qx+r=0$ are $\alpha,\ \beta,\ \gamma$. Find the equation whose roots are 3+4

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}\ ,\quad \frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{\alpha}\ ,\quad \frac{1}{\gamma}+\frac{1}{\alpha}-\frac{1}{\beta}$$

(ii) Solve the equation: $x^4+32x-60=0$.

(c) (i) Find the row reduced echelon form of 4+3

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{pmatrix}$$

(ii) Determine ker T. Where the linear mapping $T:\mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is defined by

$$T(x,\ y)=(x+2y,\ 2x+y,\ x+y),\ (x,\ y)\in \mathbb{R}^2$$

(d) Determine the condition for which the system of equations has (i) unique solution (ii) many solution (iii) infinitely many solutions 7

$$x+2y+z=1$$
$$2x+y+3z=b$$
$$x+ay+3z=b+1$$

MATHGE3

D.E. and Vector Calculus

GROUP-A

Answer *all* questions $2\times 6=12$

1. (a) What is the degree and order of the differential equations: 2

$$\frac{d^2y}{dx^2}=\left(1-\left(\frac{dy}{dx}\right)^4\right)^{1/3}$$

(b) Find the Wronskian of x, $2x+3$ and $4x+9$. What do you conclude from this Wronskian? 2

(c) Find the integrating factor of 2

$$x\cos\frac{dy}{dx}+y\,(\sin x+\cos y)=1$$

(d) Show that $f(t,\,x)=\dfrac{e^{-x}}{1+t^2}$ defined for $0<x<p$, $0<t<N$ (where N is a positive integer) satisfies Lipschitz condition with Lipschitz constant $k=p$. 2

(e) Find the general solution of $y''-5y'=0$. 2

(f) If $\dfrac{d^2x}{dt^2}+4x=0$ and $x=0$, $\dfrac{dx}{dt}=8$ at $t=0$, find x in terms of t. What is the maximum value of x? 2

GROUP-B

Answer *all* questions $5\times 4=20$

2. (a) Solve $x^2\dfrac{d^2y}{dx^2}+x\dfrac{dy}{dx}-y=x^2e^{2x}$. 5

(b) Solve by method of variation of parameter 5

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=\frac{e^{2x}}{(1+e^x)^2}$$

(c) Solve $\dfrac{d^2x}{dt^2}-\dfrac{dy}{dt}=2x+2t$ 5

$$\frac{dx}{dt}+4\frac{dy}{dt}=3y$$

(d) Solve the differential equation 5

$$(D^2 - 2D + 5)y = 12 + 25x^2 \quad , \quad D \equiv \frac{d}{dx}$$

by the method of undetermined coefficient.

GROUP-C

Answer *all* questions 7×4 = 28

3. (a) (i) Solve: $(x + y + 1)\,dx = (2x + 2y + 1)\,dy$ 3

(ii) Examine whether the differential equation $(y^2e^x + 2xy)\,dx - x^2\,dy = 0$ is exact. If exact, then solve it. 4

(b) (i) Solve: $x^3 \dfrac{d^3y}{dx^3} + x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x\log x \;\; (x > 0)$ 3

if the C.F is given by $c_1x^{-1} + c_2x + c_3x^2$.

(ii) Define Cauchy-Euler Equation. Use the substitution $x = \sinh z$ to solve the equation $(1 + x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = 4y$. 1+3

(c) (i) Solve: $y\dfrac{d^2y}{dx^2} - \left(\dfrac{dy}{dx}\right)^2 = y^2 \log_e y$ 3

(ii) If u and v are two solutions of the reduced equation of $\dfrac{d^2y}{dx^2} + P_1\dfrac{dy}{dx} + P_2y = Q$, then prove that $u\dfrac{dv}{dx} - v\dfrac{du}{dx} = Ae^{-\int P_1\,dx}$; $A \neq 0$ is a constant. 4

(d) (i) If $\dfrac{d^2x}{dt^2} + 2h\dfrac{dx}{dt} + (h^2 + p^2)x = ke^{-ht}\cos pt$, then prove that 4

$$x = c_1e^{-ht}\cos(pt + c_2) + \frac{k}{2p}te^{-ht}\sin pt$$

Show that the particular integral represents an oscillation of variable amplitude which is maximum when $ht = 1$. What is the maximum value and show that it is very large when $h \to 0$? What is the value of the amplitude when $t \to \infty$?

(ii) Solve: $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$ 3

MATHGE4

Group Theory

GROUP-A

1. Answer ***all*** the following questions: 2×6 = 12

(a) Show that in any group an element and its inverse have the same order.

(b) Is D_3 abelian?

(c) $(Z, +)$ is a sub-group of $(R, +)$. — Justify this statement.

(d) Prove that if $(ab)^2 = a^2b^2$ in a group G, then $ab = ba$ for $a, b \in G$.

(e) Give two reasons why the set of odd integers under addition is not a group.

(f) What is the order of a k-cycle $(a_1\, a_2\, a_3 \cdots\cdots a_k)$?

GROUP-B

2. Answer ***all*** the following questions: $5 \times 4 = 20$

(a) Show that symmetries of a square is a group.

(b) Let G be a group such that $a^2 = e$ for all $a \in G$ and e is the identity in G. Show that G is a commutative group.

(c) Let α and β belong to S_n. Prove that $\beta\alpha\beta^{-1}$ and α are both even or both odd.

(d) Let $H = \{a + ib : a, b \in R, \ ab \geq 0\}$. Prove or disprove that H is a subgroup of C under addition.

GROUP-C

3. Answer ***all*** the following questions: $7 \times 4 = 28$

(a) (i) For any element a and b from a group and an integer n, prove that 3+4

$$(a^{-1}ba)^n = a^{-1}b^n a$$

(ii) Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,\ b,\ c,\ d \in Z \right\}$ be a group under addition. Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G \ : \ a + b + c + d = 0 \right\}$. Prove that H is a subgroup of G. What if 0 is replaced by 1?

(b) (i) If the elements a, b and ab of a group be each of order 2, then prove that $ab = ba$. 3+4

(ii) Write all complex roots of $x^6 = 1$. Show that they form a group under the usual complex multiplication.

(c) (i) Let G be a group and let H be a subgroup of G. Prove that $N(H)$ defined by $N(H) = \{x \in G : xHx^{-1} = H\}$, the normalizer of H is a subgroup of G. 4+3

(ii) Let $(G, *)$ be a group such that $(a * b)^{-1} = a^{-1} * b^{-1}$ $\forall a,\ b \in G$, then show that G is an abelian group.

(d) (i) Let H, K be two subgroups of a group G, then show that KH is a subgroup of G if $HK = KH$. 4+3

(ii) Show that the set of all positive rational numbers do not form a group with respect to the composition '$\circ$' defined by $a \circ b = \dfrac{a}{b}$.

MATHGE5

Numerical Methods

GROUP-A

1. Answer ***all*** the questions: $2\times6 = 12$

(a) Round off the following numbers to 3 decimal places:

$$6.4455\times10^3 \quad , \quad 0.999500$$

(b) Show that the equation $x^2+\ln x=0$ has exactly one root lies in $\left[\frac{1}{3},\ 1\right]$.

(c) Compute the root of the following equation by Bisection method:

$$x^3-4x-9=0 \qquad \text{(Correct upto 2 decimal places)}$$

(d) Write down the order of convergence of the following methods:

(i) Secant method (ii) Regula-Falsi method (iii) Newton-Raphson method (iv) Jacobi-iteration method.

(e) Find the Newton's iterative formula to obtain a^{-1}.

(f) When does Gauss-Elimination method fail?

GROUP-B

2. Answer ***all*** the questions: $5\times4 = 20$

(a) If a number is correct upto n significant figures and the first significant figure of the number is k, then prove that the relative error is less than $\dfrac{1}{k\times10^{n-1}}$.

(b) Solve: $e^x-3x=0$ correct upto 3 decimal places by Newton-Raphson method.

(c) Solve the following system of equations using Gauss elimination method:

$$2x+3y-z=5$$
$$4x+4y-3z=5$$
$$-2x+3y-z=1$$

(d) Using method of iteration convert the equation $x^3+x^2-1=0$ into $x=\phi(x)$ assuming initial approximation of root is $x_0=0\cdot80$.

GROUP-C

3. Answer ***all*** the questions: $7\times4 = 28$

(a) (i) Find the number of significant figures in $V_a=1.8921$, given its relative error as 0.1×10^{-2}. 3

(ii) If $f(x,\ y,\ z)=xyz^2$ and errors in $x,\ y,\ z$ are 0.005, 0.001 and 0.002 respectively at $x=3$, $y=1$, $z=1$. Compute the maximum absolute error in evaluating f at (3, 1, 1). 4

(b) (i) Calculate $\sqrt{25\cdot 11}-\sqrt{25\cdot 1001}$ correct to three significant figures, giving necessary steps. 3

(ii) Use Gauss-Jacobi iteration method to solve the following system of equations: 4

$$3x_1+x_2+x_3=7$$
$$2x_1+x_2+5x_3=13$$
$$x_1+4x_2+x_3=9.4$$

Correct upto 2 significant figures.

(c) (i) Show that the order of convergence of Secant method is approximately 1.618. 3

(ii) Compute the root of the following equation by Regula-Falsi method: 4

$2x-3\sin x-5=0$ (Correct upto 3 decimal places)

(d) (i) Suppose the iteration $x_{n+1}=g(x_n)$ produces a sequence of numbers converging to a fixed point α. Show that this iteration method is a second order process whenever $g'(\alpha)=0$ and $g''(\alpha)\neq 0$. 3

(ii) Solve the equations by Gauss-Jordan method: 4

$$3x_1+2x_2+3x_3=18$$
$$2x_1+x_2+x_3=10$$
$$x_1+4x_2+9x_3=16$$

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