UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2021

GE-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE-I, MATHGE-II, MATHGE-III, MATHGE-IV & MATHGE-V.
The candidates are required to answer any *one* from the *five* courses.
Candidates should mention it clearly on the Answer Book.

MATHGE-I

Cal. Geo and DE.

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Find the asymptotes parallel to coordinate axes of the curves $y = \tan^{-1} x$.

(b) Evaluate: $\lim\limits_{x \to a} \left(2 - \frac{a}{x}\right)^{\tan\frac{\pi x}{2a}}$

(c) Solve: $x\,dy - y\,dx = \cos\frac{1}{x}\,dx$

(d) Evaluate: $\int\limits_0^1 \frac{x^6}{\sqrt{1-x^2}}\,dx$

(e) Find the equation of a sphere whose great circle is $x^2 + y^2 + z^2 + 10y - 4z = 8$, $x + y + z = 3$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the reduction formula for $\int (\log x)^n x^m\,dx$ and hence find the value of $\int (\log x)^2 x\,dx$. 3

(b) Find the volume of the solid obtained by revolution the region bounded by $y = x$ and $y = x^2 - 2x$ about the line $y = 5$. 4

(c) Solve: $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)e^x\sec y$ 5

3. (a) Determine the point of inflection for $y=\sin^2 x$, $0\le x\le 2\pi$. 3

(b) If $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$ is the envelope of the lines $\frac{x}{a}+\frac{y}{b}=1$, where a, b are variable parameter and c is a constant, then prove that $a^2+b^2=c^2$. 4

(c) A variable sphere passes through (0, 0, $\pm c$) and cuts the lines $y=x\tan\alpha$, $z=c$ and $y=-x\tan\alpha$, $z=-c$ in points P and P'. If $PP'=2a$ (a is a constant), show that the centre of the sphere lies on the circle $x^2+y^2=(a^2-c^2)\operatorname{cosec}^2 2\alpha$, $z=0$. 5

4. (a) Find the value of a and b if the $\lim\limits_{x\to\infty}\dfrac{a\sin^2 x+b\log\cos x}{x^4}=-\frac{1}{2}$. 3

(b) Solve: $\dfrac{dy}{dx}+2xy=x^2+y^2$ 4

(c) Reduce the canonical form $5x^2-20xy-5y^2-16x+8y-7=0$. 5

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Find the 21st derivative of $y=\dfrac{x^3}{x^2-3x-2}$. 3

(b) Trace the curve $x^2y^2=(a+y)^2(a^2-y^2)$. 4

6. (a) Solve: $(1+y^2)\,dx=(\tan^{-1}y-x)\,dy$ 3

(b) If $I_m=\dfrac{d^m}{dx^m}(x^m\log x)$, prove that $I_m=mI_{m-1}+(m-1)!$. Hence prove that $I_m=\left[\log x+1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\cdots+\frac{1}{m}\right]m!$. 4

MATHGE-II

Algebra

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Find the value of $(-i)^{1/4}$. 2

(b) Find the values of k for which the equation $x^4+4x^3-2x^2-12x+k=0$ has four real distinct roots. 2

(c) Let a, b, c be real numbers. Show that 2

$$(a+b-c)^2+(b+c-a)^2+(c+a-b)^2 \geq ab+bc+ca$$

(d) Check whether the function $f:\mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(x)=x^2+x$, $\forall x \in \mathbb{R}$ is surjective or not. 2

(e) Find the eigenvalues and eigenvectors of the matrix $7I_7$. 2

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Let z_1, z_2 be two complex numbers such that z_1/z_2 is real. Prove that the points representing z_1 and z_2 in the complex plane are collinear with the origin. 3

(b) Let $z=\cos\theta+i\sin\theta$ and m be a positive integer. Show that 5

$$(1+z)^m+\left(1+\frac{1}{z}\right)^m=2^{m+1}\cos^m\frac{\theta}{2}\cos\frac{m\theta}{2}$$

(c) Show that $\tan 4\theta=\dfrac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$. 4

3. (a) Apply Descartes' Rule of signs to find the nature of the roots of the equation $x^6-3x^2-2x+3=0$. 3

(b) If $ax \equiv ay \pmod{m}$ where a is prime to m then prove that $x \equiv y \pmod{m}$. 4

(c) Let α, β, γ be the roots of the equation $x^3+2x^2-3x+1=0$. Find the equations whose roots are $\alpha\beta-\gamma^2$, $\beta\gamma-\alpha^2$, $\gamma\alpha-\beta^2$ and deduce the condition that the roots of the given equation may be in geometric progression. 4+1

4. (a) Prove that the relation $\rho=\{(a, b): a \equiv b \pmod{3}\,;\ a, b \in \mathbb{Z}\}$ is an equivalence relation on the set of integers $\mathbb{Z}$. Also, find the sets $A=\{a:(a, 1) \in \rho\}$, $B=\{a:(a, 2) \in \rho\}$ and $C=\{a:(a, 3) \in \rho\}$ where $a \in \mathbb{Z}$, show that $\mathbb{Z}=A \cup B \cup C$ and $A \cap B=B \cap C=C \cap A=\phi$. 3+4

(b) Reduce the following matrix: 4+1

$$\begin{bmatrix} 0 & 0 & 2 & 2 & 0 \\ 1 & 3 & 2 & 4 & 1 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

into row reduced echelon form and hence find its rank.

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Prove that $1!\cdot 3!\cdot 5!\cdot \cdots\cdots (2n-1)! > (n!)^n$. 3

(b) If $x_0, x_1, \ldots\ldots, x_{n-1}$ are the roots of $x^n - 1 = 0$, then prove that $(x_0 - x_1)(x_0 - x_2)\ldots\ldots(x_0 - x_{n-1}) = n$. 4

6. (a) Use Cayley-Hamilton Theorem to find A^{2021}, where $A = \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix}$. 2

(b) Determine the conditions for which the system of equations 5

$$x + y + z = b$$
$$2x + y + 3z = b + 1$$
$$5x + 2y + az = b^2$$

admits of (i) no solution, (ii) unique solution and (iii) many solutions.

MATHGE-III

Differential Equation and Vector Calculus

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Show that $\sin 3x$, $\cos 3x$ is not a linearly independent solution of $y'' + 9y = 0$.

(b) Solve the equation: $\dfrac{dx}{dt} = -wy$ and $\dfrac{dy}{dt} = wx$.

(c) Find the particular integral of $\dfrac{d^2y}{dx^2} + y = \cos x$.

(d) If $\boldsymbol{r} = 3t\boldsymbol{i} + 3t^2\boldsymbol{j} + 2t^3\boldsymbol{k}$, then find $\dfrac{d\boldsymbol{r}}{dt} \times \dfrac{d^2\boldsymbol{r}}{dt^2}$.

(e) Given $\vec{r} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$. Show that $\left[\dfrac{d\vec{r}}{dt}\;\dfrac{d^2\vec{r}}{dt^2}\;\dfrac{d^3\vec{r}}{dt^3}\right] = a^2 b$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Solve, using the method of variation parameters, the following differential equation: 6

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y = \log x$$

(b) Evaluate the line integral $\int_C \boldsymbol{F} \cdot \boldsymbol{dr}$ along the curve $C : x^2 + y^2 = 1,\ z = 2$ in the positive direction from $A(1, 0, 2)$ to $B(0, 1, 2)$ where 6

$$\boldsymbol{F} = (y + xz^2)\boldsymbol{i} + (z - y)\boldsymbol{j} + (xy - z)\boldsymbol{k}$$

3. (a) Solve, using the method of undetermined coefficients, the equation 6

$$(D^2 - 3D)y = x + e^x \sin x\ ,\ \left(D \equiv \frac{d}{dx}\right)$$

(b) Evaluate $\left(\iint \vec{F} \cdot \hat{n}\right) ds$, where $\vec{F} = yz\hat{i} + x\hat{j} + z\hat{k}$ and S is the surface of the cylinder $x^2 + y^2 = 4$ included in the first octant between $z = 0$ and $z = 2$. 6

4. (a) Solve the differential equation $\frac{d^2y}{dx^2} + a^2 y = \sec ax$ with the symbolic operator D. 6

(b) If $\vec{F}(x, y, z) = r^n(\vec{\alpha} \times \vec{r})$, then prove that $\vec{F}$ is solenoidal, where $\vec{\alpha}$ is a constant vector, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. 6

GROUP-C

Answer *all* the questions from the following 7×2=14

5. Find the complete solution of the given differential equation 7

$$(D^3 + 1)y = e^{2x} \sin x + e^{\frac{x}{2}} \sin\left(\frac{\sqrt{3}x}{2}\right)$$

6. What is an irrotational vector? Find the constants a, b, c so that the vector 7

$$\vec{F} = (x + 2y + az)\vec{i} + (bx - 3y - z)\vec{j} + (4x + cy + 2z)\vec{k}$$

is irrotational. Hence, find the potential function V.

MATHGE-IV

Group Theory

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) If $G = (\mathbb{R}, +)$ and $H = (\mathbb{Z}, +)$, then how many distinct left cosets of H in G are there?

(b) Let $H := \{a + bi : a, b \in \mathbb{R},\ ab \geq 0\}$. Prove or disprove that H is a subgroup of $\mathbb{C}$ of all complex numbers under addition.

(c) Find an isomorphism from the group of integers $(\mathbb{Z}, +)$ to the group of even integers $(2\mathbb{Z}, +)$.

(d) "Every abelian group is cyclic" — Check whether the statement is true or false with proper justification.

(e) Check whether the set of rational numbers $\mathbb{Q}$ forms a group with respect to multiplication.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the number of elements of order 5 in the group $(\mathbb{Z}_{30}, +)$. 5

(b) Give an example of a group G and it's subgroup H such that $[G:H]=2$. 5

(c) Let $(\mathbb{Z}, +)$ be the group of integers and $N=\{3n \mid n\in\mathbb{Z}\}$. Prove that N is a normal subgroup of $\mathbb{Z}$. 2

3. Let G be a group and $g\in G$. Define a map $\varphi_g : G\to G$ by $\varphi_g(x) := gxg^{-1}$ for all $x\in G$. Show that φ_g is an isomorphism from G onto itself. 6+6

Further, define $\mathcal{L}(G) := \{\varphi_g : g\in G\}$. Show that $\mathcal{L}(G)$ forms a group under composition of functions.

4. (a) If G is a cyclic group with only one generator, prove that either $o(G)=1$ or $o(G)=2$. 4

(b) Give an example of two subgroups H, K of a group which are not normal but HK is a subgroup. 4

(c) Suppose a group G has a subgroup of order n. Show that the intersection of all subgroups of G of order n is a normal subgroup of G. 4

GROUP-C

Answer *all* the questions from the following 7×2=14

5. (a) Prove that a cyclic group G of order n is isomorphic to the multiplicative group consisting of all n^{th} roots of unity. 5

(b) Show that for any group G, $G/\{e\}\cong G$. 2

6. (a) Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. 4

(b) Write down the cyclic decomposition of the following permutation as a product of disjoint cycles 3

$$\sigma=\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 5 & 6 & 1 & 3 & 2 & 8 & 7 \end{pmatrix}$$

MATHGE-V

Numerical Methods

GROUP-A

1. Answer ***all*** the questions from the following: 2×5=10

(a) Given that $u = \dfrac{5xy^2}{z^3}$ find the relative error at $x = y = z = 1$ when the errors in each of x, y, z is 0.001.

(b) Write the sufficient condition for convergence of Gauss-Seidel iteration method.

(c) The equation $x^2 + ax + b = 0$ has two real roots α and β. Show that $x_{k+1} = -(ax_k + b)/x_k$ is convergent near $x = \alpha$ if $|\alpha| > |\beta|$.

(d) Calculate $\sqrt{25.11} - \sqrt{25.1001}$, correct up to three significant figures.

(e) Show that $\Delta^n[ke^{ax}] = k(e^{ah} - 1)^n e^{ax}$.

GROUP-B

Answer *all* the questions from the following 12×3=36

2. (a) Find the polynomial of degree three relevant to the following data by Lagrange's formula: 6

x	-1	0	1	2
$f(x)$	1	1	1	-3

(b) Evaluate $\displaystyle\int_2^3 \frac{dx}{1+2x}$ by Trapezoidal Rule taking $n = 10$ and compare the result with the exact value of the integration. 6

3. (a) Show that the Bisection method converges linearly. 6

(b) Determine a, b and c such that the formula 6

$$\int_0^h f(x)\,dx = h\left[af(0) + bf\left(\frac{h}{3}\right) + cf(h)\right]$$

is exact for polynomial of as high order as possible and determine the order of truncation error.

4. (a) Use Picard's method to solve the differential equation $\dfrac{dy}{dx} = x^2 + \dfrac{y}{2}$ at $x = 0.5$, correct to two decimal places, given that $y = 1$ when $x = 0$. 6

(b) Solve the following system of linear equations by Gauss-Seidel method: 6

$$20x + 5y - 2z = 14$$
$$3x + 10y + z = 17$$
$$x - 4y + 10z = 23$$

GROUP-C

Answer *all* the questions from the following 7×2=14

5. Using modified Euler's method, find $y(4.4)$ by taking $h = 0.2$, from the following differential equation: 7

$$5x\frac{dy}{dx} + y^2 - 2 = 0 \ , \quad y(4) = 1$$

6. If the third order differences of a function $f(x)$ are constants and 7

$$\int_{-1}^{1} f(x)\,dx = k\left[f(0) + f\left(\frac{1}{\sqrt{2}}\right) + f\left(-\frac{1}{\sqrt{2}}\right)\right],$$

then find the value of k.

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