UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2021

GE-MATHEMATICS

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE4-I, MATHGE4-II, MATHGE4-III, MATHGE4-IV & MATHGE4-V.
The candidates are required to answer any *one* from the *five* courses.
Candidates should mention it clearly on the Answer Book.

MATHGE4-I

CAL. GEO. AND DE.

GROUP-A

1. Answer the following questions: 2×5 = 10

(a) Let the axes of a given coordinate system be changed to lines making an angle α with the corresponding old axes, without changing the origin. Let the coordinate of a point A with respect to the first system be $(\sqrt{3}, 1)$ and its coordinates with respect to the second system be (2, 0). Then find the value of α.

(b) Solve: $\frac{dy}{dx} + \tan y \tan x = \cos x \sec y$

(c) Find the area of the lemniscates $r^2 = a^2 \cos 2\theta$.

(d) Find the envelope of the family of straight lines $y = mx + \sqrt{a^2m^2 + b^2}$, m being the parameter.

(e) Determine the conic represented by $3x^2 + 10xy + 3y^2 - 2x - 14y - 13 = 0$.

GROUP-B

Answer *all* the following questions 12×3= 36

2. (a) If by a rotation of coordinate axes the expressions $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is changed to 5

$$a'x^2 + 2h'xy + b'y^2 + 2g'x + 2f'y + c' = 0$$

show that $a' + b' = a + b$ and $a'b' - h'^2 = ab - h^2$.

(b) Test whether the equation $(x+y)^2 dx - (y^2 - 2xy - x^2)\, dy = 0$ is exact and hence solve. 3

(c) If $I_n = \int_0^{\pi/2} x \sin^n x \, dx$, $n > 1$ show that $I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$. 4

Hence evaluate $\int_0^{\pi/2} x \sin^5 x \, dx$.

3. (a) Show that the envelope of the family of circles whose centres lie on the parabola $y^2 = 4ax$ and which pass through its vertex is the curve $y^2(2a+x)+x^3 = 0$. 4

(b) Evaluate: $\lim_{x\to\infty}\left\{x-\sqrt[n]{(x-a_1)(x-a_2)\cdots(x-a_n)}\right\}$ 4

(c) If $y = a(x+\sqrt{x^2-1})^n + b(x+\sqrt{x^2-1})^n$ prove that $(x^2-1)y_{n+2} + x(2n+1)y_{n+1} = 0$. 4

4. (a) Solve: $(x^2+y^2+x)\,dx + xy\,dy = 0$ 3

(b) If a plane passes through a fixed point (f, g, h) and cuts the axes at the points P, Q, R respectively then show that the locus of the centre of the sphere passing through the origin and the points P, Q, R is $\frac{f}{x}+\frac{g}{y}+\frac{h}{z} = 2$. 5

(c) Find the volume of the solid generated by revolving the cycloid 4

$$x = a(\theta+\sin\theta)\ ,\quad y = a(1+\cos\theta)$$

about its base.

GROUP-C

Answer following questions 7×2 = 14

5. (a) Show that the locus of the point of intersection of perpendicular tangents to the conic $l/r = 1+e\cos\theta$ is $r^2(1-e^2)+2ler\cos\theta-2l^2 = 0$. 5

(b) Find the range of values of x for which $y = x^4-6x^3+12x^2+5x+7$ is concave upwards. 2

6. (a) Show that the points of intersection of the curve 4

$$2y^3-2x^2y-4xy^2+4x^3-14xy+6y^2+4x^2+6y+1=0$$

and its asymptotes lie on the line $8x+2y+1=0$.

(b) Find the trace the curve of $y^2(2a-x) = x^3$. 3

MATHGE4-II

ALGEBRA

GROUP-A

1. Answer the following questions: 2×5 = 10

(a) Find $\mathrm{Sin}^{-1}(2i)$ and $\sin^{-1}(2i)$. 2

(b) If α, β, γ be the roots of the equation $x^3+px^2+qx+r=0$, find the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$. 2

(c) Let a, b, c be positive real numbers. Prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3$. 2

(d) Give an example of a symmetric relation which is neither reflexive nor transitive. Justify your answer. 2

(e) Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} 3 & 0 \\ 2 & 3 \end{bmatrix}$. 2

GROUP-B

Answer the following questions 12×3 =36

2. (a) Let z be a variable complex number such that the ratio $\frac{z-i}{z+1}$ is purely imaginary. Show that the point z lies on a circle in the complex plane. 3

(b) If $z = \cos\theta + i\sin\theta$ and $\varsigma = \cos\varphi + i\sin\varphi$, prove that $\frac{z^m}{\varsigma^n} + \frac{\varsigma^n}{z^m} = 2\cos(m\theta - n\varphi)$, where m, n are integers. 4

(c) Find the general values and the principal value of i^{x+iy}, where x, y are real. Also check the principal value for x is an even and an odd integer. 3+2

3. (a) Determine the nature of the roots of the equation $x^6 + 2x^5 - 5x^4 - 3x^2 - 2x + 19 = 0$, by using the Descartes' Rule of signs. 3

(b) Let α, β, γ be the roots of the equation $x^3 - 5x^2 + 3x - 1 = 0$. Find the equations whose roots are $\alpha - \beta\gamma^2$, $\beta - \gamma\alpha^2$, $\gamma - \alpha\beta^2$ and deduce the condition that the roots of the given equation may be in geometric progression. 4+1

(c) Using the Cayley-Hamilton Theorem, find the inverse of the matrix 4

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix}$$

4. (a) Let $f : A \to B$ be a mapping. A relation ρ is defined on A by "$x\rho y$ if and only if $f(x) = f(y)$, $x, y \in A$". Show that ρ is an equivalence relation on A. 4

(b) If $a \equiv b \pmod{m}$ and $a \equiv b \pmod{n}$ where $\gcd(m, n) = 1$, prove that $a \equiv b \pmod{mn}$. 3

(c) Reduce the following matrix: 4+1

$$\begin{bmatrix} 7 & 0 & 2 & 1 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 2 & 8 & 4 & -4 & 2 \\ 3 & 5 & 1 & 3 & 6 \end{bmatrix}$$

into row reduced echelon form and hence find its rank.

GROUP-C

Answer the following questions 7×2 = 14

5. (a) If $n \in \mathbb{Z}$, then show that $2n + 1 \le 2^n$ for all $n \ge 3$. 3

(b) Solve the equation $x^4 + 3x^3 + 5x^2 + 4x + 2 = 0$. 4

6. (a) Let x be an eigen-vector corresponding to the eigen-value λ of a matrix M and n be a positive integer. Show that x is an eigen-vector of M^n corresponding to λ^n. 2

(b) Determine the conditions for which the following system of equations has (i) no solution, (ii) unique solution and (iii) many solution: 5

$$x+2y+z=1$$
$$2x+y+3z=k$$
$$x+\alpha y+z=k+1$$

MATHGE4-III

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

Answer *all* the following questions $2\times 5=10$

1. (a) Find the Wronskian of x and xe^x. Hence, conclude whether or not these are linearly independent.

(b) Solve $x^2y''+xy'-9y=0$, given that $y=x^3$ is a solution.

(c) Find $\nabla(\log|r|)$, where $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$.

(d) If vectors $\mathbf{a}$ and $\mathbf{b}$ are irrotational, show that $\mathbf{a}\times\mathbf{b}$ is solenoidal.

(e) What are the order and degree of the differential equation of the family of curves $y^2=2c(x+\sqrt{c})$.

GROUP-B

Answer *all* the following questions $12\times 3=36$

2. (a) By computing the appropriate Lipschitz constant, show that the following function 6

$$f(x,\ y)=x^2\cos y+y\sin^2 x$$

satisfy Lipschitz condition on the domain $D:|x|\le 1,\ |y|<\infty$ of xy plane.

(b) Solve the differential equation $\dfrac{d^2y}{dx^2}+a^2y=\sec ax$ with the symbolic operator D. 6

3. (a) Apply the method of variation of parameters to solve the following differential equation 6

$$(x-1)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=(x-1)^2$$

(b) If $\vec{a}=(2y+3)\hat{i}+zx\hat{j}+(yz-x)\hat{k}$, evaluate $\int\vec{a}\cdot\overrightarrow{dr}$. Where the integration is over the curve $x=2t^2,\ \ y=t,\ \ z=t^3$ from $t=0$ to $t=1$. 6

4. (a) Show that $\mathbf{F}=2xyz^2\mathbf{i}+(x^2z^2+z\cos yz)\mathbf{j}+(2x^2yz+y\cos yz)\mathbf{k}$ is a conservative field of force. Find the scalar potential Ø. Find also the work done in moving an object in this field from the point (0, 0, 1) to $(1,\frac{\pi}{4},2)$. 6

(b) Use method of undetermined coefficient to solve the following differential equation: 6

$$(D^2-4D+4)y=x^3e^{2x}+xe^{2x},\quad \text{where } D\equiv\frac{d}{dx}$$

GROUP-C

Answer *all* the following questions 7×2 = 14

5. (a) If $y_1(x)$ and $y_2(x)$ are any two solutions of $a_0(x)y''(x)+a_1(x)y'(x)+a_2(x)y(x)=0$, show that the linear combination $c_1y_1(x)+c_2y_2(x)$, where c_1 and c_2 are constants, is also a solution of the given equation. 3

(b) Solve: $\dfrac{dx}{y+z}=\dfrac{dy}{z+x}=\dfrac{dz}{x+y}$ 4

6. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$ wehre $\mathbf{F}=(y^2+z^2)\mathbf{i}+(z^2+x^2)\mathbf{j}+(x^2+y^2)\mathbf{k}$ from (0, 0, 0) to (2, 4, 8) and C is the curve given by $\mathbf{r}=t\mathbf{i}+t^2\mathbf{j}+t^3\mathbf{k}$. 7

MATHGE4-IV

GROUP THEORY

GROUP-A

Answer *all* the following questions 2×5 = 10

1. Find the order of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{pmatrix}$$

in the permutation group S_6.

2. Let $G=(\mathbb{Z}_8,+)$ and $f:G\to G$ be a homomorphism defined by
$$f([x])=2[x]$$
Find ker f.

3. For each positive integer $n\geq 2$, let U_n denotes the set of all positive integers that are smaller than n and relatively prime to n. Consider the group U_n, equipped with multiplication modulo n. Show that for any integer $n>2$, there exist at least two elements in U_n that satisfy $x^2=1$.

4. Show that if two right cosets Ha and Hb of a group G be distinct, then the left cosets $a^{-1}H$ and $b^{-1}H$ are also distinct.

5. Prove that the group of integers $(\mathbb{Z},+)$ is not isomorphic to the group of rationals $(\mathbb{Q},+)$.

GROUP-B

Answer *all* the following questions 12×3=36

6. (a) Show that the set $\mathbb{Z}_1$ of all odd integers forms a group with respect to the binary operation * defined on $\mathbb{Z}_1$ by $a*b:=a+b-3$ for $a,b\in\mathbb{Z}_1$. 4

(b) For $\theta \in \mathbb{R}$, let $\mathbb{R}_\theta$ denotes the rotation matrix 8

$$\mathbb{R}_\theta := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Show that the set $SL_2(\mathbb{R}) := \{\mathbb{R}_\theta : 0 \leq \theta < 2\pi\}$ forms a group under matrix multiplication. Find two non-trivial finite subgroups of $SL_2(\mathbb{R})$.

7. (a) Prove that there does not exist an onto homomorphism from the group $(\mathbb{Z}_6, +)$ to the group $(\mathbb{Z}_4, +)$. 4

(b) Show that $(\mathbb{Q}, +)$ cannot be isomorphic to $(\mathbb{Q}^*, \cdot)$, where $\mathbb{Q}^* = \mathbb{Q} - \{0\}$. 4

(c) Find all the subgroups of D_4. 4

8. (a) Suppose the table below is a group table. Fill in the blank entries. 7

*	e	a	b	c	d
e	e				
a		b			e
b		c	d	e	
c		d		a	b
d					

Is this group cyclic? — Justify.

(b) Let $\mathbb{R}$ denote the group of all real numbers under addition and $\mathbb{Z}$ denote the group of all integers under addition. Let C denote the unit circle in the complex plane, i.e., $C := \{z \in \mathbb{C} : |z| = 1\}$. 5

Consider C as the group under usual complex multiplication. Exhibit a surjective homomorphism $\varphi : \mathbb{R} \to \mathbb{C}$ with $\ker\varphi = \mathbb{Z}$. Hence conclude that $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{C}$.

GROUP-C

Answer *all* the following questions 7×2 = 14

9. (a) If H is a subgroup of a group G such that $(aH)(Hb)$ for any $a, b \in G$ is either a left or a right coset of H in G, then prove that H is normal. 3

(b) Show that homomorphic image of a finite group is finite. 4

10.(a) For a positive integer $n \geq 2$, exhibit a homomorphism from the symmetric group S_n to the group $\mathbb{Z}_2$ of integers modulo 2. What is the kernel of this homomorphism? 4

(b) In N is a normal subgroup of a group G, and the order of the quotient group G/N is m, then show that for each element $x \in G$, $x^m \in N$. 3

MATHGE4-V

NUMERICAL METHODS

GROUP-A

Answer *all* the following questions 2×5 = 10

1. (a) Define the degree of precision of a quadrature formula. What is the degree of precision of the Trapezoidal rule?

(b) Evaluate: $\left(\frac{\Delta^2}{E}\right)x^3$

(c) Round off the numbers to three decimal places: 6.445×10^3, 0.999500.

(d) If $f(x)=4\cos x+6x$, find the relative percentage error in $f(x)$ for $x=0$, if the error is $x=0.005$.

(e) Give the geometrical significance of $f'(x)$ in approximation of the root of the equation $f(x)=0$ by Newton-Raphson method.

GROUP-B

Answer *all* the following questions 12×3= 36

2. (a) In usual notation, show that $\Delta^m\left(\frac{1}{x}\right)=\frac{(-1)^m m!\,h^m}{x(x+h)(x+2h)\ldots(x+mh)}$ 6

(b) A certain function *f*, defined on the interval (0, 1) is such that $f(0)=0$, $f\left(\frac{1}{2}\right)=-1$, $f(1)=0$. Find the quadratic polynomial $p(x)$ which agrees with $f(x)$ for $x=0,\ 1/2,\ 1$. 6

If $\left|\frac{d^3f}{dx^3}\right|\le 1$ for $0\le x\le 1$, show that $|f(x)-p(x)|\le\frac{1}{12}$ for $0\le x\le 1$.

3. (a) Solve the following system of linear equations by Gauss elimination process: 6

$$a+2b+c=0\ ;\ 2a+2b+3c=3\ ;\ -a-3b=2$$

(b) Using modified Euler's method, find $y(0.2)$ for the differential equation $y'=\frac{x-y}{2}$, $y(0)=1$ with step length 0.1. 6

4. (a) Find the smallest positive root of the equation $\cos x-5x+1=0$ using the iterative method, correct up to two decimal places. 6

(b) Prove that $f(x_k,\ x_{k-1},\ x_{k-2},\cdots\cdots\cdots,x_{k-n})=\frac{\nabla^n f(x_k)}{h^n n!}$, where the arguments are equispaced and ∇ being a backward difference operator. Hence show that 6

$$f(x_n,\ x_{n-1},\ x_{n-2},\cdots\cdots\cdots,x_0)=\frac{\nabla^n f(x_n)}{h^n n!}$$

GROUP-C

Answer *all* the following questions 7×2 = 14

5. Using Lagrange interpolation, find a polynomial $P(x)$ of degree < 4 satisfying 7

$$P(1)=1,\quad P(2)=4,\quad P(3)=1,\quad P(4)=5$$

6. Evaluate $\int_1^4 \frac{\log_e(1+0.5x+x^2)}{0.5+x}\,dx$, by Trapezoidal rule, correct up to 6 decimal places, taking 13 ordinates. 7

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