UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2021

CC9-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-I

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer *all* the following questions 2×5 = 10

1. (a) Prove that in a field F, $a^2 = b^2$ implies either $a = b$ or $a = -b$.

   (b) $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined by $T(x, y, z) = (x + y + z,\ y + z,\ z)$ is a linear mapping. Show that T is non-singular.

   (c) Find the units in the ring of $(\mathbb{Z}_8, +, \cdot)$.

   (d) Prove that the set $S = \{(a,\ 3b) :\ a, b \in \mathbb{Z}\}$ is a subring of the ring $\mathbb{Z} \times \mathbb{Z}$.

   (e) Find the co-ordinate vector of (0, 3, 1) in $\mathbb{R}^3$ relative to the basis $\{(1, 1, 0),\ (1, 0, 1),\ (0, 1, 1)\}$.

GROUP-B

10×3=30

2. (a) Show that $H = \left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} :\ \alpha,\ \beta \in \mathbb{C} \right\}$ is a subring of $M_2(\mathbb{C})$. Is H a field? 4

   (b) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} :\ a, b \in \mathbb{Z} \right\}$ be the ring and $\varphi : R \longrightarrow \mathbb{Z}$ is defined by $\varphi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) = a - b$. Show that φ is a ring homomorphism and find $\ker \varphi$. 4

   (c) Check whether the ring $C[0, 1]$ of real valued continuous functions on [0, 1] is an integral domain or not. Justify your answer. 2

3. (a) Extend the set {(1, 0, 1, 0), (0, 1, 1, 1)} to obtain a basis of the vector space $\mathbb{R}^4$. 4

(b) Find the dimension of the subspace $S \cap T$ of $\mathbb{R}^4$, where $S = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 0\}$ and $T = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y - z + w = 0\}$. 4

(c) Prove that $\mathbb{Z}/2\mathbb{Z}$ is isomorphic to $5\mathbb{Z}/10\mathbb{Z}$. 2

4. (a) Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ is defined by $T(x, y, z) = (2x + y - z,\ y + 4z,\ x - y + 3z)$ for all $(x, y, z) \in \mathbb{R}^3$. Find the matrix of T and the inverse of T relative to the ordered basis $\{(1, 0, 1), (1, 1, 0), (0, 1, 1)\}$ of $\mathbb{R}^3$. 5

(b) Let $\varphi : \mathbb{Z} \longrightarrow \mathbb{Z}_5$ be defined by $\varphi(x) =$ the remainder of x (mod 5) for each $x \in \mathbb{Z}$. Show that φ is an onto homomorphism and find $\mathbb{Z}/\ker\varphi$. 3

(c) Find all the ring homomorphisms from $\mathbb{Z}_{15}$ to $\mathbb{Z}_3$. 2

GROUP-C

10×1 =10

5. (a) Let $T : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ be a map defined by 2+4

$$T(a, b, c, d) = (a - b + c + d,\ a + 2c - d,\ a + b + 3c - 3d)$$

(i) Show that T is a linear transformation.

(ii) Find the basis and dimension of range space and null space of T.

(b) Let B be an ordered basis for a 2-dimensional vector space V over a field F and T a linear mapping on V. If the matrix representation of T with respect to B is $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then show that $T^2 - (a + d)T + (ad - bc)I = 0$. 4

GROUP-D

5×2 = 10

6. Let $(R, +, \cdot)$ be a ring. Define the operations $\oplus$ and $\odot$ on R by 5

$$r \oplus s = r + s + 1 \quad \text{and} \quad r \odot s = r \cdot s + r + s$$

Show that (i) $(R, \oplus, \odot)$ is a ring. (ii) $(R, \oplus, \odot)$ is isomorphic to the ring $(R, +, \cdot)$.

7. (a) Show that the set $I = \{a + bi\sqrt{5} \mid a, b \in \mathbb{Z},\ a - b \text{ is even}\}$ is an ideal of the ring $\mathbb{Z}[i\sqrt{5}]$. 3

(b) Let R be a commutative ring with unity of prime characterization p. Show that the map $f : R \to R$ given by $f(a) = a^p$ for all $a \in R$ is a ring homomorphism. 2

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