UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 6th Semester Examination, 2021

CC13-MATHEMATICS

RING THEORY AND LINEAR ALGEBRA-II

Full Marks: 60

ASSIGNMENT

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer *all* questions from the following $2\times5 = 10$

1. Let $\boldsymbol{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be the basis of C^3 defined by $\alpha_1 = (1,\ 0,\ -1)$, $\alpha_2 = (1,\ 1,\ 1)$, $\alpha_3 = (2,\ 2,\ 0)$. Find the dual basis of $\boldsymbol{B}$.

2. Show that $\sqrt{-3}$ is a prime element in the integral domain $\mathbb{Z}[\sqrt{-3}]$.

3. Find the orthogonal complement of $W = \text{span}\,\{(1,\ 1,\ 1)\}$ in the Euclidean space $\mathbb{R}^3$ with standard inner product.

4. Let $(|)$ denotes the standard inner product on $\mathbb{R}^2$. Let $\alpha = (2,\ 1)$, $\beta = (1, -1)$. If μ is a vector such that $(\alpha\,|\,\mu) = 3$, $(\beta\,|\,\mu) = 2$, then find μ.

5. Show that $1-i$ is irreducible in $\mathbb{Z}[i]$.

GROUP-B

Answer *all* questions from the following $10\times3 = 30$

6. (a) Use Gram-Schmidt process to obtain an orthogonal basis from the basis $\{(1,\ 0,\ 1), (1,\ 1,\ 1), (1,\ 3,\ 4)\}$ of Euclidean space $\mathbb{R}^3$ with standard inner product. 4+3+3

   (b) Let $\mathbb{R}^3$ be a Euclidean space with standard inner product and $T : V \to V$ be defined by $T(x,\ y,\ z) = (x+2y,\ x-z,\ x+3y-2z)$. Find T^*, adjoint of T.

   (c) Find an orthonormal basis of the row space of the matrix $\begin{pmatrix} 1 & 1 & 1 & 0 \\ 2 & 3 & 1 & 1 \\ 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix}$.

7. (a) Find all the maximal and prime ideals of $\mathbb{Z}_{10}$. 3+3+4

(b) Let D be a Euclidean domain with Euclidean valuation ν. If $a|b$ and $\nu(a)=\nu(b)$, prove that a and b are associates in D.

(c) Is the integral domain $\mathbb{Z}[\sqrt{-5}]=\{a+b\sqrt{-5}:a,b\in\mathbb{Z}\}$, a unique factorization domain? Justify your answer.

8. (a) Find a 3×3 matrix for which the minimal polynomial is x^2. 5+5

(b) Let T be a linear operator on $\mathbb{R}^4$ which is represented in the standard basis by the matrix.

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{pmatrix}$$

Under what conditions, T is diagonalizable?

GROUP-C

Answer *all* questions from the following 5×2 = 10

9. (a) Find the eigen values and corresponding eigenspace of the matrix kI_5. Generalize the result for the matrix kI_n. 4+1

(b) Show that the matrix $\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$ is not diagonalizable.

10. If N_1, N_2 be any two normal operators such that either permutes with the adjoint of the other, then prove that N_1+N_2 and N_1N_2 are normal. 5

GROUP-D

Answer *all* questions from the following 5×2 = 10

11. Find the minimal polynomial of the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$. 5

12.(a) Use Cayley-Hamilton theorem to find A^{70}, where $A=\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. 2+3

(b) Let R be a ring of all real valued continuous functions defined on [0, 1] and $M=\{f(x)\in R: f(1/5)=0\}$. Prove that M is a maximal ideal of R.

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