'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 1st Semester Examination, 2021

CC1-MATHEMATICS

CALCULUS, GEOMETRY AND DIFFERENTIAL EQUATION

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions: 3×4 = 12

(a) If $\lim_{x\to 0} \dfrac{e^x - ae^{x\cos x}}{x - \sin x}$ exists finitely, find the value of a. Then find the value of the limit. 3

(b) Show that the curve $y = e^{-x^2}$ has points of inflexion at $x = \pm\dfrac{1}{\sqrt{2}}$. 3

(c) If $I_n = \displaystyle\int_0^{\pi/2} x^n \sin x \, dx$ and $n > 1$, show that $I_n + n(n-1)I_{n-2} = n(\pi/2)^{n-1}$. 3

(d) Find the area in the first quadrant included between the parabola $y^2 = bx$ and the circle $x^2 + y^2 = 2bx$. 3

(e) Show that the length of the focal chord of the conic $\dfrac{l}{r} = 1 - e\cos\theta$ which is inclined to the initial line at an angle α is $\dfrac{2l}{(1 - e^2\cos^2\alpha)}$. 3

(f) Prove that the differential equation of all circles touching the y-axis at the origin is $(y^2 - x^2)\,dx - 2xy\,dy = 0$. 3

GROUP-B

2. Answer any ***four*** questions: 6×4 = 24

(a) Find the envelope of the curves $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, where the two parameters a and b are connected by the relation $a + b = c$, c being a fixed constant. 6

(b) Prove that the volume of the solid obtained by revolving the lemniscate $r^2 = a^2\cos 2\theta$ about the initial line is $\dfrac{1}{2}\pi a^3\left\{\dfrac{1}{\sqrt{2}}\log(\sqrt{2}+1) - \dfrac{1}{3}\right\}$. 6

(c) Find the asymptotes of the curve $(x^2 - y^2)(x + 2y + 1) + (x + y + 1) = 0$. 6

(d) If PSP' and QSQ' are two perpendicular focal chords of a conic, prove that $\dfrac{1}{SP \cdot SP'} + \dfrac{1}{SQ \cdot SQ'} =$ a constant. 6

(e) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by 6

$$(x^2 + y^2 + z^2)\,(x^{-2} + y^{-2} + z^{-2}) = 4r^2$$

(f) Solve $x\cos\frac{y}{x}(y\,dx + x\,dy) = y\sin\frac{y}{x}(x\,dy - y\,dx)$. 6

GROUP-C

Answer any *two* questions 12×2=24

3. (a) Obtain the limit of $\lim\limits_{x\to\infty}\frac{e^{-2x}(\cos x + 2\sin x)}{e^{-x}(\cos x + \sin x)}$. 6

(b) Show that the pedal equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with respect to the centre as pole is $\frac{a^2b^2}{p^2} = a^2 + b^2 - r^2$. 6

4. (a) Show that the volume of the solid formed by revolving one loop of the curve $r^2 = a^2\cos 2\theta$ about the line $\theta = \frac{\pi}{2}$ is $\frac{\pi^3 a^3}{4\sqrt{2}}$. 6

(b) If $2\frac{dr}{d\theta} + r\tan\theta = \frac{1}{r\cos\theta}$ and $r = 1$ when $\theta = 0$. Show that $r = 4\sqrt{2}$ when $\theta = \frac{\pi}{4}$. 6

5. (a) Find the equations of the parabolas passing through the common points of $x^2 + 6xy - y^2 + 2x - 3y - 5 = 0$ and $2x^2 - 8xy + 3y^2 + 2y - 1 = 0$. 4

(b) Find the equation of the sphere for which the circle $x^2 + y^2 + z^2 + 2x - 4y + 2z + 5 = 0$, $x - 2y + 3z + 1 = 0$ is a great circle. 4

(c) Show that the plane $12y + z - 2x - 16 = 0$ intersects the paraboloid $x^2 - 4y^2 = 2z$ in two generators $\frac{x}{2} = \frac{y-2}{1} = \frac{z+8}{-8}$ and $\frac{x}{2} = \frac{y-4}{-1} = \frac{z+32}{16}$. 4

6. (a) Transform the differential equation $\frac{dy}{dx} + \frac{y}{x}\log y = \frac{y}{x^2}(\log y)^2$ into a linear form and then solve it. 6

(b) Show that a differential equation of the form 6

$$[y + x\,f(x^2 + y^2)]\,dx + [y\,f(x^2 + y^2) - x]\,dy = 0$$ is not exact.

Show that $\frac{1}{x^2 + y^2}$ is an integrating factor of an equation of this form.

Hence solve, $[y + x(x^2 + y^2)^2]\,dx + [y(x^2 + y^2)^2 - x]\,dy = 0$.

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