'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 5th Semester Examination, 2021

DSE-P2-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains DSE2A and DSE2B. Candidates are required to answer any *one* from the *two* DSE2 courses and they should mention it clearly on the Answer Book.

DSE2A

NUMBER THEORY

GROUP-A

1. Answer any ***four*** questions from the following: $3\times4 = 12$

(a) Use Euclidean Algorithm to obtain integers x and y, such that $\gcd(119, 272) = 119x + 272y$. 3

(b) Prove that, $7 \mid (111^{333} + 333^{111})$. 3

(c) Determine whether the following quadratic congruence has a solution or not: 3

$$x^2 \equiv 2 \pmod{71}$$

(d) Solve $34x \equiv 60 \pmod{98}$. 3

(e) Show that $a^{21} \equiv a \pmod{15}$ for all $a \in \mathbb{Z}$. 3

(f) If p_n is the n^{th} prime, prove that $p_n \leq 2^{2^{n-1}}$. 3

GROUP-B

Answer any *four* questions from the following $6\times4 = 24$

2. Prove that the Diophantine Equation $x^4 + y^4 = z^2$ has no solution in integers. 6

3. A certain integer between 1 and 1200 leaves the reminder 1, 2, 6 when divided by 9, 11, 13 respectively. What is the integer? 6

4. For $n > 2$ prove that there exists a prime p such that $n < p < n!$. If p_n be the n^{th} prime number, show that $\frac{1}{p_1} + \frac{1}{p_2} + \ldots\ldots + \frac{1}{p_n}$ is not an integer. 3+3

5. Find all the positive integral solutions of the following Diophantine equation: 6

$$142x + 20y = 1000$$

6. Show that the equation $6x^2 + 5x + 1 = 0$ has no solution in integers but for every prime p the equation $6x^2 + 5x + 1 \equiv 0 \pmod{p}$ has a solution. 6

7. (a) Evaluate the Legendre symbol $(3658/12703)$. 3

(b) Show that $5^{38} \equiv 4 \pmod{11}$. 3

GROUP-C

Answer any *two* questions from the following 12×2 =24

8. Prove that there are infinitely many prime numbers and among these infinitely many are of the form $(4k-1)$ for some $k \in \mathbb{Z}$. Show that for a prime p and $n > 2$ the terms of the A.P. $p,\ p+d,\ p+2d,\ \ldots\ldots,\ p+(n-1)d$ will be primes if the common difference d is divisible by every prime $q < n$. 6+6

9. $\alpha,\ \beta \neq 0$ are two Gaussian integers prove that there exist integers $\mu,\ \rho \in Z[i]$ such that $\alpha = \mu\beta + \rho$, with $|\rho| < |\beta|$. Show that μ and ρ are not unique. 8+4

10. Show that all solutions of the Pythagorean equation $x^2 + y^2 = z^2$ satisfying $\gcd(x,\ y,\ z) = 1,\ 2\,|\,x,\ x,\ y,\ z > 0$ are given by $x = 2st,\ \ y = s^2 - t^2,\ \ z = s^2 + t^2$ where $s,\ t \in \mathbb{Z}$ with $s > t$ $\gcd(s,\ t) = 1$ and $s \not\equiv t \pmod{2}$. Show that the radius of the incircle of a Pythagorean triangle is always an integer. 6+6

11.(a) Prove that no prime p of the form $4k+3$ is a sum of two squares. 3

(b) Prove that an odd prime p is expressible as a sum of two square if and only if $p \equiv 1 \pmod{4}$. 9

DSE2B

MECHANICS

GROUP-A

1. Answer any ***four*** questions from the following: 3×4 = 12

(a) If the resistance per unit mass is $g(\frac{u}{v})^2$, prove that $\frac{du}{ds} = \frac{-g}{v^2}u$, $\frac{d\psi}{ds} = \frac{g}{u^2}\cos^3\psi$, where u is the horizontal component of velocity.

(b) A hemisphere rests in equilibrium on a sphere of equal radius. Show that the equilibrium is unstable when the curved surface of the hemisphere rests on the sphere.

(c) The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$, show that the inclination to AB of one of the principal axes at A is $\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2-b^2)}$.

(d) The velocities of a particle along and perpendicular to the radius vector from a fixed origin are λr^2 and $\mu\theta^2$. Show that the path is $\frac{\lambda}{\theta}=\frac{\mu}{2r^2}+c$.

(e) Prove that the resultant turn about the astatic centre through the same angle.

(f) Write down the Kepler's laws of planetary motion.

GROUP-B

Answer any *four* questions from the following $6\times4=24$

2. A uniform rod of length $2a$ and weight W rests on a rough horizontal plane (coefficient of friction λ), its weight being uniformly distributed along its length. If the rod is just about to move under the action of force P applied perpendicular to the rod at a distance c from the centre, show that $\frac{a}{\sqrt{a^2+c^2}-c}=\frac{\lambda W}{P}$. 6

3. Prove that any system of forces acting on a rigid body can be reduced to a single force and a couple whose axis lies along the line of action of the force. 6

4. A particle moves with a central acceleration $\mu\left(r+\frac{c^4}{r^3}\right)$, being projected from an apse at a distance c with a velocity $2\sqrt{\mu}\,c$, prove that its path is $r^2(2+\cos\sqrt{3}\phi)=3c^2$. 6

5. A particle moves freely in a parabolic path given by $y^2=4ax$ under a force which is always perpendicular to its axis. Find the law of force. 6

6. An attracting force, varying as the distance, acts on a particle initially at rest at a distance a. Show that if V be the velocity when the particle is at a distance x and V' be the velocity of the same particle when the resistance of air is taken into account, then $V'=V\left[1-\frac{1}{3}k\frac{(2a+x)(a-x)}{a+x}\right]$ nearly, the resistance of the air being given to be k times the square of the velocity per unit mass, where k is very small. 6

7. A uniform square plane $ABCD$ of mass M and side $2a$ lies on a smooth horizontal plane. It is struck at A by a particle of mass M' moving with velocity in the direction AB, the particle remaining attached to the plate. Determine the subsequent motion of the system and show that its angular velocity is $\frac{M'}{M+4M'}=\frac{3V}{2a}$. 6

GROUP-C

Answer any *two* questions from the following 12×2 =24

8. (a) A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it describes in time t is $\frac{1}{\mu V}[(\sqrt{1+2\mu V^2 t})-1]$. 4

(b) If ω be the angular velocity of a planet at the nearest of the major axis, prove that its period is $\frac{2\pi}{\omega}=\sqrt{\left\{\frac{1+e}{(1-e)^3}\right\}}$. 3

(c) A particle subject to a central force per unit mass equal to $\mu\{2(a^2+b^2)u^2-3a^2b^2u^7\}$ is projected at a distance a with a velocity $\frac{\sqrt{\mu}}{a}$ in a direction of right angle to the distance; show that the path is the curve $r^2=a^2\cos^2\theta+b^2\sin^2\theta$. 5

9. (a) A rocket is fired from the earth surface with speed V at an angle α to the radius through the point of projection. Show that the rocket's subsequent greatest distance from the earth's centre is the larger root of the quadratic equation, 6

$$\left(V^2-\frac{2GM}{R}\right)r^2+2GMr-R^2V^2\sin^2\alpha=0$$

If $V^2<\frac{2GM}{R}$, where R is the radius and M be the mass of the earth and G is the gravitational constant. Deduce that the escape velocity is independent of α.

(b) Forces P, Q, R act along any three mutually perpendicular generators of the same system of the surface $x^2+y^2=2(z^2+a^2)$, the positive direction of the forces being towards the same side of the plane xy. Prove that the pitch of the equivalent wrench is $2a\frac{(PQ+QR+RP)}{P^2+Q^2+R^2}$. 6

10. (a) Two equal forces act along each of the straight lines $\frac{x\mp a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{\mp b\cos\theta}=\frac{z}{c}$; show that their central axis must, for all values of θ, lie on the surface $y\left(\frac{x}{z}+\frac{z}{x}\right)=b\left(\frac{a}{c}+\frac{c}{a}\right)$. 4

(b) Write down the equation of motion of a particle moving in a central orbit under a central force F and deduce the differential equation of the orbit in the form $\frac{h^2}{P^3}\frac{dP}{dr}=F$, where the symbols have the usual meaning. 4

(c) Show that the equilibrium is stable or unstable according as $\frac{\cos\alpha}{h}\gtrless\frac{1}{\rho_1}+\frac{1}{\rho_2}$. Where α is the angle between the common normal and the vertical at the point of contact, h is the height of c.g. from the point of contact and ρ_1, ρ_2 are the radii of curvatures at the point of contact. 4

11.(a) Find the equation of the central axis of a given system of forces. 4

(b) An artificial satellite is circling round the earth at height of 700 km from the surface. Calculate the horizontal velocity of the satellite. 3

(Radius of earth = 6300 km, $g = 981$ cm/s^2).

(c) AB is a uniform rod of length $6a$ and weight W which can turn freely about a fixed point in its length distance $2a$ from A, AC and BC are light strings of length $5a$ attached to a partial C of weight W'. Show that if W is less than $2W'$, there will be stable equilibrium with AB inclined to the horizontal at an angle $\tan^{-1}\left(\dfrac{W+W'}{4W'}\right)$. 5

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