'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 2nd Semester Examination, 2022

GE1-P2-MATHEMATICS

Time Allotted: 2 Hours | Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

The question paper contains MATHGE-I, MATHGE-II, MATHGE-III, MATHGE-IV & MATHGE-V.
The candidates are required to answer any *one* from the *five* courses.
Candidates should mention it clearly on the Answer Book.

MATHGE-I

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GROUP-A

Answer any *four* questions from the following $3\times 4 = 12$

1. Evaluate: $\int\limits_0^{\pi/4} \tan^5 x\, dx$. 3

2. Find the equation of the curve $3x^2 + 3y^2 + 6x - 18y - 14 = 0$ referred to parallel axes through the point (–1, 3). 3

3. Determine the concavity and the inflexion points of $f(x) = x^3 + 3x^2 - 9x + 8$. 3

4. Transform the following equation into Cartesian form $r = 2\sin 3\theta$. 3

5. Solve: $(2x\cos y + 3x^2 y)\,dx + (x^3 - x^2 \sin y - y)\,dy = 0 \quad ; \quad y(0) = 2$ 3

6. Find the asymptote of the curve $x^2 y^2 = a^2(x^2 + y^2)$. 3

GROUP-B

Answer any *four* questions from the following $6\times 4 = 24$

7. Find the trace of $y^2(2a - x) = x^3$. 6

8. Show that $\int \tan^2 x \sec^4 x\, dx = \frac{1}{5}\tan^5 x + \frac{1}{3}\tan^3 x$. 6

9. If $y = e^{3\sin^{-1} x}$, then show that $(1-x^2)y_{n+2} - (2n+1)x\, y_{n+1} - (n^2+9)\, y_n = 0$. 6

10. Locate the vertex and focus of the parabola $x^2 - 4x - 12y - 15 = 0$, write the equation of the directrix axis and tangent at the vertex. 6

11. Find the envelope of $x^2 \sin\alpha + y^2 \cos\alpha = a^2$, where α is a parameter. 6

12. Solve: $xy - \frac{dy}{dx} = y^3 e^{-x^2}$ 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) If $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$, $|x| < 1$, show that $(1-x^2)y_{n+2} - (2n+3)x\,y_{n+1} - (n+1)^2 y_n = 0$. 6

(b) If $I_{m,n} = \int_0^{\pi/2} \cos^m x \sin nx\,dx$, then show that $I_{m,n} = \frac{1}{m+n} + \frac{m}{m+n} I_{m-1,\,n-1}$. 6

14.(a) Determine the value of α and β for which $\lim_{x\to 0} \frac{\sin 3x + \alpha \sin 2x + \beta \sin x}{x^5}$ exists and find the limit. 6

(b) Show that the perpendicular from the origin on the generators of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$ lie on the cone $\left(\frac{x}{a} - \frac{y}{b}\right)(ax - by) + 2z^2 = 0$. 6

15.(a) Solve: $\frac{dy}{dx} + x \sin xy = e^x y^n$ 6

(b) Reduce the equation $\sin y \frac{dy}{dx} = \cos x (2\cos y - \sin^2 x)$ to a linear equation and hence solve it. 6

16.(a) Reduce the equation $x^2 + y^2 + z^2 - 2xy - 2yz + 2zx + x - 4y + z + 1 = 0$ to its canonical form and determine the type of the quadric represented by it. 6

(b) Determine the concavity and the inflection points of the function $f(x) = 3x^4 - 4x^2 + 1$. 6

MATHGE-II

ALGEBRA

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. If k be a positive integer then prove that $\gcd(ka, kb) = k \gcd(a, b)$.

2. Determine k so that the set S is linearly independent in $\mathbb{R}^3$; where $S = \{(1, 2, 1), (k, 3, 1), (2, k, 0)\}$.

3. If a, b, c be three positive real numbers, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3$.

4. If λ be an eigenvalue of $A\in M_n(\mathbb{R})$, prove that λ^m is an eigenvalue of A^m.

5. Find $\operatorname{mod} z$ and $\arg z$, where $z=i^i$.

6. Solve the equation $x^4-x^3+2x^2-2x+4=0$, one root being $1+i$.

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. (a) Determine the rank of the matrix $\begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}$. 3

(b) Consider a matrix A whose eigenvalues are 1, –1 and 3. Then find trace (A^3-3A^2). 3

8. (a) If n be a positive integer, prove that $\frac{1}{\sqrt{4n+1}}<\frac{3\cdot 7\cdot 11\cdot\,.....\,\cdot(4n-1)}{5\cdot 9\cdot 13\cdot\,.....\,\cdot(4n+1)}<\sqrt{\frac{3}{4n+3}}$. 3

(b) If α be a multiple root of order 3 of the equation $x^4+bx^2+cx+d=0$, $(d\neq 0)$. Show that $\alpha=-\frac{8d}{3c}$. 3

9. (a) Prove that $3^{2n}-8n-1$ is divisible by 64 for any non-negative integer n. 3

(b) Consider the function $f:\mathbb{R}\longrightarrow(-1,1)$, defined by $f(x)=\frac{x}{1+|x|}$ for all $x\in\mathbb{R}$. Prove that f is bijective. 3

10.(a) Let $f:A\to B$ and $g:B\to C$ be two functions. Then prove that if $g\circ f$ is bijective then f is injective and g is surjective. 3

(b) What is the residue when 11^{40} is divided by 8? 3

11.(a) Find the product of all the values of $(1+i)^{4/5}$. 4

(b) State the Cauchy-Schwartz inequality. 2

12.(a) For what values of 'a' the following system of equation is consistent? 4

$$x-y+z=1$$
$$x+2y+4z=a$$
$$x+4y+6z=a^2$$

(b) Give an example of a binary relation which is reflexive and transitive but not symmetric. 2

GROUP-C

Answer any *two* questions from the following $12\times 2 = 24$

13.(a) State and prove the fundamental theorem of equivalence relation. 6

(b) Let n be a positive integer and $\mathbb{Z}_n$ denote the set of all congruence classes of $\mathbb{Z}$ modulo n. Prove that the number of elements of $\mathbb{Z}_n$ is finite. 6

14.(a) Prove the following identities: 3+3

(i) $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$

(ii) $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(b) Let n be a positive integer and a, b and c are integers such that $a \neq 0$. Then prove that $ab \equiv ac \pmod{n}$ if and only if $b \equiv c \left(\bmod \dfrac{n}{\gcd(a,\, n)}\right)$. 6

15.(a) State Cayley-Hamilton theorem for matrices. Use it to find A^{-1} and A^{50}, where $A = \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}$. 3+3

(b) Find the rank of the following matrices of order n: 6

(i) Nilpotent matrix

(ii) Idempotent matrix and

(iii) Involuntary matrix.

16.(a) Consider the mapping $f : \mathbb{Z}_0^+ \times \mathbb{Z} \longrightarrow \mathbb{Z}$, defined by $f(m,\, n) = 2^m(2n+1)$ for all $(m,\, n) \in \mathbb{Z}_0^+ \times \mathbb{Z}$. Prove that f is injective but not surjective. Here $\mathbb{Z}_0^+$ denotes the set of all non-negative integers. 4

(b) Prove that composition of mappings is associative. Show by a counter example that composition of mapping is not commutative. 2+3

(c) Apply Descarte's rule of signs to find the positive and negative roots of the following equation: 3

$$4x^3 - 8x^2 - 19x + 26 = 0$$

MATHGE-III

DIFFERENTIAL EQUATION AND VECTOR CALCULUS

GROUP-A

Answer any *four* questions from the following $3\times 4 = 12$

1. Show that $f(t,\, x) = \dfrac{e^{-x}}{1+t^2}$ defined for $0 < x < p$, $0 < t < N$, where N is a positive integer, satisfies Lipschitz condition with Lipschitz constant $K = p$. 3

2. Find the Wronskian of $\{1-x,\ 1+x,\ 1-3x\}$. 3

3. Examine continuity of the vector valued function $\vec{r}=t^3\hat{i}+e^t\hat{j}+\frac{1}{t+3}\hat{k}$ at $t=-3$. 3

4. Find the directional derivative of $\phi=xy^2z+4x^2z$ at (–1, 1, 2) in the direction $2\hat{i}+\hat{j}-2\hat{k}$. 3

5. Find the particular integral of $\frac{d^2y}{dx^2}+4y=\sin 2x$. 3

6. Solve: $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$, with $x(0)=0$, $\frac{dx(0)}{dt}=0$. 3

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. If y_1 and y_2 are solutions of $y''+x^2y'+(1-x)y=0$ such that $y_1(0)=0$, $y_2(0)=1$, $y_1'(0)=1$, $y_2'(0)=-1$, then find the Wronskian $W(y_1,\ y_2)$. 6

8. Solve by the method of variation of parameters $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=\frac{e^{-x}}{x^2}$. 6

9. Solve: $(D^3-D^2-6D)\,y=(1+x+x^2)e^x$, where $D\equiv\frac{d}{dx}$. 6

10. Solve: $4x'+9y'+44x+49y=t$ 6
$3x'+7y'+34x+38y=e^t$

11. If $\vec{A}=3xy\hat{i}-5z\hat{j}+10x\hat{k}$, then evaluate $\int_C \vec{A}.d\vec{r}$ along the curve C given by $x=t^2+1$, $y=2t^2$, $z=t^3$ from $t=1$ to $t=2$. 6

12. Show that the vector field given by $\vec{A}=(y^2+z^3)\hat{i}+(2xy-5z)\hat{j}+(3xz^2-5y)\hat{k}$ is conservative and find the scalar point function for the field. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Find the general solution of $t^2y''-3ty'+7y=0$, $t>0$. 6

(b) Solve: $(D-1)^2(D^2+1)\,y=e^x+\sin^2\frac{x}{2}$ 6

14.(a) If $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$, then show that $\vec{\nabla}\cdot\left\{\frac{f(r)}{r}\vec{r}\right\}=\frac{1}{r^2}\frac{d}{dr}\{r^2f(r)\}$. 6

(b) Prove that $\vec{\nabla}\cdot(\vec{F}\times\vec{G})=\vec{G}\cdot(\vec{\nabla}\times\vec{F})-\vec{F}\cdot(\vec{\nabla}\times\vec{G})$, where $\vec{F}$ and $\vec{G}$ are vector point functions. 6

15.(a) Solve by method of undetermined coefficients. 6

$$(D^2 - 3D)y = x + e^x \sin x\ ,\quad D \equiv \frac{d}{dx}$$

(b) Solve: $\dfrac{dx}{dt} + \dfrac{2}{t}(x - y) = 1$; $\dfrac{dy}{dt} + \dfrac{1}{t}(x + 5y) = t$ 6

16.(a) If $\vec{F} = \phi\,\text{grad}\phi$, then show that $\vec{F}\cdot\text{curl}\,\vec{F} = 0$. 6

(b) Solve: $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10x + \dfrac{10}{x}$ 6

MATHGE-IV

GROUP THEORY

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. Define normal subgroup of a group. 3
2. Find all cyclic subgroups of the group $(\mathbb{Z}_7, +)$. 3
3. Show that identity and inverse of an element in a group G are unique. 3
4. Show that (6 5 4 3 1 2) is an odd permutation. Find the images of 3 and 4 if $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 1 & & & 3 \end{pmatrix}$ be an even permutation. 1+2
5. Show that any infinite cyclic group is isomorphic to $(\mathbb{Z}, +)$. 3
6. Show that centre of a group G is a subgroup of G. 3

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. Define subgroup of a group G. Let H, K be two subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$. 1+5
8. Show that every cyclic group is commutative. Is the converse true? Justify your answer. 4+2
9. Let H be a subgroup of G and let $a \in G$. Then show that $aH = H$ if and only if $a \in H$. 6
10. State and prove Lagrange's theorem. 6
11. Find all the homomorphisms from $\mathbb{Z}_{20} \longrightarrow \mathbb{Z}_8$. How many of them are onto? 5+1
12. Find all the subgroups of $\mathbb{Z}/12\mathbb{Z}$. Find the subgroup lattice of D_4. 3+3

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Show that $G=\{1, -1, i, -i\}$ forms an abelian group with respect to multiplication. 6

(b) In a group $(G, *)$, if $b^5 = e_G$ and $b*a*b^{-1}=a^2$, $\forall\, a, b \in G$, find the order of 'a'. 6

14.(a) Let $\phi:(G, \circ) \to (G', *)$ be a homomorphism. Prove that $\ker\phi$ is a normal subgroup of G. 4

(b) Prove that order of $U(n)$ is even for $n \geq 3$. 2

(c) Find all elements of order 5 in $(\mathbb{Z}_{40}, +)$. Find all the cyclic subgroups of $(\mathbb{Z}_9, +)$. 6

15.(a) If H be a subgroup of a cyclic group G, then prove that the quotient group G/H is cyclic. 6

(b) Prove that every permutation on a finite set is either a cycle or a product of disjoint cycles. 6

16. Let H be a subgroup of G. Then show that the following conditions are equivalent:

(a) H is a normal subgroup of G. 4

(b) $gHg^{-1} \subseteq H$, $\forall\, g \in G$ 4

(c) $gHg^{-1} = H$, $\forall\, g \in G$. 4

MATHGE-V

NUMERICAL METHODS

GROUP-A

Answer any *four* questions from the following 3×4 = 12

1. Show that $\Delta \log f(x) = \log\left[1+\dfrac{\Delta f(x)}{f(x)}\right]$.

2. Find the number of significant figures in:

(i) $V_A = 11.2461$ given its absolute error as 0.25×10^{-2}.

(ii) $V_T = 1.5923$ given its relative error as 0.1×10^{-3}.

3. State the condition for convergence of Gauss-Seidel method for solving a system of equations. Are they necessary and sufficient?

4. Given the set of values of $y = f(x)$:

x	2	4	6	8	10
y	5	10	17	29	50

Form the diagonal difference table and find $\Delta^2 f(6)$.

5. What is the geometric representation of the Newton-Raphson method?

6. Find the function whose first difference is e^x taking the step size $h=1$.

GROUP-B

Answer any *four* questions from the following 6×4 = 24

7. The equation $x^2+ax+b=0$ has two real roots α, β. Show that the iteration method $x_{k+1}=-\dfrac{ax_k+b}{x_k}$ is convergent near $x=\alpha$, if $|\alpha|>|\beta|$. 6

8. What is interpolation? Establish Lagrange's polynomial interpolation formula. 1+5

9. Using the method of Newton-Raphson, find the root of $x^3-8x-4=0$ which lies between 3 and 4, correct upto 4 decimal places. 6

10. Complete the following table: 6

x	10	15	20	25	30	35
$f(x)$	19.97	21.51	—	23.52	24.65	—

11. Use Euler's method, solve the following problem for $x=0.1$ by taking $h=0.02$. 6

$$\frac{dy}{dx}=\frac{y-x}{y+x} \quad \text{with} \quad y(0)=1$$

12. Calculate the approximate value of $\int_0^{\pi/2}\sin x\,dx$, by Trapezoidal rule using 11 ordinates. 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Given the following table: 3+3

x	0	5	10	15	20
$f(x)$	1.0	1.6	3.8	8.2	15.4

Construct the difference table and compute $f(21)$ by Newton's Backward formula.

(b) Solve the system by Gauss-Jacobi iteration method: 6

$$x+y+4z=9$$
$$8x-3y+2z=20$$
$$4x+11y-z=33$$

14.(a) Evaluate $\int_0^{\pi/2}\sqrt{1-0.162\sin^2\phi}\,d\phi$, by Simpson's $\frac{1}{3}$rd rule, correct upto four decimal places, taking six sub-interval. 6

(b) Show that Bisection method converges linearly. 6

15.(a) Use Picard's method to compute $y(0.1)$ from the differential equation $\frac{dy}{dx}=1+xy$ given $y=1$, where $x=0$. 6

(b) Define the operator Δ. Prove that $\Delta^n\left(\frac{1}{x}\right)=\frac{(-1)^n\, n!}{x(x+1)(x+2).....(x+m)}$. 1+5

16.(a) Establish Newton's forward interpolation formula. 6

(b) Use Gauss-elimination method to solve the following: 6

$$-10x_1+6x_2+3x_3+100=0$$
$$6x_1-5x_2+5x_3+100=0$$
$$3x_1+6x_2-10x_3+100=0$$

Correct upto three significant figures.

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