'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2022

CC8-MATHEMATICS

MULTIVARIATE CALCULUS

Time Allotted: 2 Hours　　　　Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

Answer any *four* questions from the following 　　$3\times4=12$

1. Let $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x-y} & , \quad x\neq y\\ 0 & , \quad x=y\end{cases}$ 　3

   Show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

2. Find the directional derivative of $f(x,y)=x^2+y^2$ at (1, 1) in the direction of unit vector $\beta=\left(\dfrac{2}{\sqrt{5}},\dfrac{-1}{\sqrt{5}}\right)$. 　3

3. If $V=f(xyz)$, prove that $x\dfrac{\partial V}{\partial x}=y\dfrac{\partial V}{\partial y}=z\dfrac{\partial V}{\partial z}$. 　3

4. Evaluate $\int_0^1\int_{x^2}^{x} xy\,dx\,dy$ by changing the order of integration. 　3

5. Show that the vector $\vec{V}=(4xy-z^3)\hat{i}+2x^2\hat{j}-3xz^2\hat{k}$ is irrotational. 　3

6. By using double integration formula find the area of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. 　3

GROUP-B

Answer any *four* questions from the following 　　$6\times4=24$

7. Let $f:D\to\mathbb{R},\ D\subseteq\mathbb{R}^2$ and $(a,b)\in D$. Let one of the partial derivatives f_x and f_y exists and the other is continuous at (a, b). Prove that f is differentiable at (a,b). 　6

8. If $u=\log r$ and $r^2=x^2+y^2+z^2$, prove that 　6

$$r^2\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=1$$

9. Prove that $\int_0^1 dx\int_0^1\dfrac{x^2-y^2}{(x^2+y^2)^2}dy\neq\int_0^1 dy\int_0^1\dfrac{x^2-y^2}{(x^2+y^2)^2}dx$. 　6

10. State Stoke's theorem. Verify Stoke's theorem for 6

$$\vec{A} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k},$$

where the surface S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.

11. Evaluate $\iint (1-x-y)^{l-1}x^{m-1}y^{n-1}dx\,dy$ taken over the interior of the triangle formed by the lines $x=0,\ y=0$; $x+y=1$; where l, m, n being all positive. 6

12. Define a conservative vector field. Prove that a vector field $\vec{F}$ is conservative over a region, if and only if $\oint \vec{F}\cdot d\vec{r}$ be zero along any closed curve in the region. 1+5 = 6

GROUP-C

Answer any *two* questions from the following 12×2 = 24

13.(a) Show that $\iint \{2a^2 - 2a(x+y) - (x^2+y^2)\}\,dx\,dy = 8\pi a^4$, the region of integration being the circle $x^2+y^2+2a(x+y)=2a^2$. 6

(b) Let f be a differentiable function of two independent variables u, v and u, v be differentiable functions of one independent variable x. Prove that f is a differentiable function of x and $\dfrac{df}{dx} = \dfrac{\partial f}{\partial u}\cdot\dfrac{du}{dx} + \dfrac{\partial f}{\partial v}\cdot\dfrac{dv}{dx}$. 6

14.(a) Let $f(x,\ y) = \begin{cases} x^2\tan^{-1}\dfrac{y}{x} - y^2\tan^{-1}\dfrac{x}{y} & ,\quad xy \neq 0 \\ 0 & ,\quad xy = 0 \end{cases}$ 6

Show that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

(b) Evaluate $\iint\limits_R \dfrac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}}\,dx\,dy$ the field of integration being R, the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. 6

15.(a) Use Divergence theorem to evaluate $\iint\limits_S \vec{F}\cdot d\vec{S}$ where S is the surface enclosing the cylinder $x^2+y^2=4$, $z=0$, $z=3$ and $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$. 6

(b) Apply Green's theorem in the plane to evaluate 6

$$\oint_C \{(y - \sin x)\,dx + \cos x\,dy\}$$

where C is the triangle enclosed by the lines $y=0$, $x=\pi$, $y=\dfrac{2x}{\pi}$.

16.(a) Prove that the necessary and sufficient condition that the vector field defined by the vector point function $\vec{F}$ with continuous derivatives be conservative is that $\text{curl}\,\vec{F} = \vec{\nabla}\times\vec{F} = 0$. 2+4 = 6

(b) Use Stoke's theorem to prove that 6

(i) curl grad $\phi = 0$, where ϕ is a scalar function.

(ii) div curl $\vec{F} = 0$, where F is a vector field.

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