'समानो मन्त्रः समितिः समानी'

UNIVERSITY OF NORTH BENGAL

B.Sc. Honours 4th Semester Examination, 2022

CC10-MATHEMATICS

METRIC SPACES AND COMPLEX THEORY

Time Allotted: 2 Hours — Full Marks: 60

The figures in the margin indicate full marks.
All symbols are of usual significance.

GROUP-A

1. Answer any ***four*** questions from the following: $3\times 4 = 12$

(a) Prove that in any metric space (X, d) every closed sphere is a closed set. 3

(b) Show that $f(z) = |z|^2$ is nowhere differentiable except $z = 0$. 3

(c) Suppose X is a metric space and $\{x_n\}$ is a convergent sequence in X with limit α. Show that the subset $\{x_n : n \in \mathbb{N}\} \cup \{\alpha\}$ of X is compact. 3

(d) Find the value of $\int_C \frac{z^2-4}{z^2+4}\, dz$, where $C: |z-i| = 2$. 3

(e) Prove that the real line $\mathbb{R}$ is not compact. 3

(f) Show that $\int_C (z-z_0)^n\, dz = \begin{cases} 2\pi i & , \text{ if } n=-1 \\ 0 & , \text{ if } n \neq -1 \end{cases}$, 3

where C is the circle with centre z_0 and radius $r > 0$ traversed in the anti-clockwise direction.

GROUP-B

2. Answer any ***four*** questions from the following: $6\times 4 = 24$

(a) Prove that a compact metric space is complete. Is the converse true? Justify your answer. $4+2 = 6$

(b) Prove that the function 6

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, \quad z \neq 0$$
$$= 0, \quad z = 0$$

is continuous and that CR equations are satisfied at the origin but $f'(0)$ does not exist.

(c) (i) Show that if two connected sets are not separated, then their union is connected. $4+2 = 6$

(ii) Show that every totally bounded metric space is bounded.

(d) (i) Evaluate $\int_C \frac{\cosh(\pi z)}{z(z^2+1)}dz$ using Cauchy's integral formula, where $C: |z|=2$. 3+3 = 6

(ii) Expand $\frac{z^2-1}{(z+2)(z+3)}$ in the region $|z|>3$.

(e) (i) Prove that every non-constant polynomial $p(z)=a_0+a_1z+\cdots+a_nz^n$, has at least one-zero in $\mathbb{C}$. Where $a_j, j=0,1,2,\cdots,n$ are complex constants and $a_n \neq 0$. 4+2 = 6

(ii) Evaluate $\int_C \frac{e^z}{z^2-2z}\,dz$, where $C: |z|=4$.

(f) Let (X, d) and (Y, d') be two metric spaces. Show that a function $f: X \to Y$ is continuous iff for any $x \in X$ and for all sequence $\{x_n\}$ converges to x in (X, d), the sequence $\{f(x_n)\}$ converges to $f(x)$ in (Y, d'). 6

GROUP-C

3. Answer any ***two*** questions from the following: 12×2 = 24

(a) (i) If $f(z)$ is differentiable in a region G and $|f(z)|$ is constant in G, then show that $f(z)$ is constant in G. 3+6+3 = 12

(ii) State and prove Cauchy's integral formula for disk.

(iii) Prove that every compact metric space is separable.

(b) (i) If $f(z)$ is an analytic function of z, show that 3+6+3 = 12

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|\operatorname{Re} f(z)|^2 = 2|f'(z)|^2$$

(ii) Prove that every compact metric space is complete and totally bounded.

(iii) Let A be a subset of a metric space (X, d) and $A \neq \phi$. Define $d(x, A)=\inf\{d(x, a): a \in A\}$, $x \in X$. Show that the map $f: X \to \mathbb{R}$ defined by $f(x)=d(x, A)$ is uniformly continuous over X.

(c) (i) Prove that a necessary and sufficient condition that a function $f(z)=u(x, y)+iv(x, y)$ tend to $l=\alpha+i\beta$ as $z=x+iy$ tend to $z_0=a+ib$ is that $\lim_{(x, y)\to(a,b)} u(x, y)=\alpha$ and $\lim_{(x,y)\to(a,b)} v(x, y)=\beta$. 4+4+4 = 12

(ii) Prove that if an entire function f is bounded for all values of z. Then f is constant.

(iii) Let f be an entire function with $f(0)=1$, $f(1)=2$ and $f'(0)=0$. If there exists $M>0$ such that $|f''(z)| \leq M$ for all $z \in \mathbb{C}$, then find $f(z)$.

(d) (i) Show that the real line ($\mathbb{R}$, d) is connected, when d is the usual metric. 6+6 = 12

(ii) Show that a metric space is compact iff every collection of closed sets in X having finite intersection property has non-empty intersection.

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